COMPLETING THE SURVIVOR DERIVATIVES MARKET:
A GENERAL PRICING FRAMEWORK

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Abstract
Survivorship is a risk of considerable importance to developed economies. Survivor derivatives are in their early stages and manage a risk which is arguably more serious than that managed by credit derivatives. This paper takes the approach developed by Dowd et al. [2006], Olivier and Jeffery [2004], Smith [2005] and Cairns [2007] for pricing survivor swaps and shows its application to the pricing of other forms of linear survivor derivatives, such as forwards, basis swaps, forward swaps and futures. It then shows how a recent option pricing model set out by Dawson et al. [2009] can be used to price survivor options such as survivor swaptions, caps, floors and combined option products. It concludes by considering applications of these products to a pension fund that wishes to hedge its survivorship risks.

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1. INTRODUCTION

A new global capital market, the Life Market, is developing (see, e.g., Blake et al. [2008]) and ‘survivor pools’ (or ‘longevity pools’ or ‘mortality pools’ depending on how one views them) are on their way to becoming the first major new asset class of the twenty-first century. This process began with the securitization of insurance company life and annuity books (see, e.g., Millette et al. [2002], Cowley and Cummins [2005] and Lin and Cox [2005]). But with investment banks, such as Goldman Sachs, entering the growing market in pension plan buyouts in the UK, in particular, it is only a matter of time before full trading of ‘survivor pools’ in the capital markets begins.¹ Recent developments in this market include the launch of the LifeMetrics Index in March 2007, the first derivative transaction, a \( q \)-forward transaction, based on this index in January 2008 (see Coughlan et al. (2007) and Grene [2008]), and the first survivor swap executed in the capital markets between Canada Life and a group of ILS and other investors in July 2008 with JPMorgan as the intermediary.

However, the future growth and success of this market depends on participants having the right tools to price and hedge the risks involved, and there is a rapidly growing literature that addresses these issues. The present paper seeks to contribute to that literature by

¹ Dunbar [2006]. On 22 March 2007, the Institutional Life Markets Association (ILMA) was established in New York by Bear Stearns (bought by JPMorgan in 2008), Credit Suisse, Goldman Sachs, Mizuho International, UBS and West LB AG. The aim is to ‘encourage best practices and growth of the mortality and longevity related marketplace’.
setting out a general framework for pricing survivor derivatives. This framework has two principal components, one applicable to linear derivatives such as swaps, forwards and futures, and the other applicable to survivor options. The former is a generalization of the swap-pricing model first set out by Dowd et al. [2006], which they applied to simple vanilla survivor swaps. We show that this approach is capable of considerable generalization and can be used to price a range of other linear survivor derivatives. The second component is the application of the option-pricing model set out by Dawson et al. [2009] to the pricing of survivor options such as survivor swaptions. This is a very simple model based on a normally distributed underlying, and it can be applied to survivor options in which the underlying is the swap premium or price, because the latter is approximately normal. Having set out this framework and shown how it can be used to price survivor derivatives, we then illustrate their possible applications to the various survivorship hedging alternatives available to a pension fund, such as hedging with survivor swaps, forwards, swaptions, caps and collars.

This paper is organized as follows. Section 2 sets out a framework to price survivor derivatives in an incomplete market setting, and uses it to price vanilla survivor swaps. Section 3 then uses this framework to price a range of other linear survivor derivatives: these include survivor forwards, forward survivor swaps, survivor basis swaps and survivor futures contracts. Section 4 extends the pricing framework to price survivor swaptions, caps and floors, making use of an option pricing formula set out in Dawson et al. [2009]. Section 5 gives a number of hedging applications of our pricing framework, and section 6 concludes.
2. PRICING VANILLA SURVIVOR SWAPS

2.1 A model of aggregate longevity risk

It is convenient if we begin by outlining an illustrative model of aggregate longevity risk. Note to begin with that we are concerned in this paper only with the risk-neutral or pricing probability (as is done, e.g., in Cairns et al. [2006] and Cairns [2007])\(^2\), and not with ‘real-world’ probabilities as such. Let \( p(s, t, u) \) be the (risk-neutral) probability based on information available at \( s \) that an individual who is alive at \( t \) will survive to \( u \) (referred to as the forward survival probability by Cairns et al. 2006). Our initial estimate of the (risk neutral) survival probability to \( u \) is therefore \( p(0, 0, u) \), whereas the equivalent true probability to \( u \) is \( p(u, 0, u) \). It follows that for each \( s = 1, \ldots, t \), we get

\[
p(s, t - 1, t) = p(s - 1, t - 1, t) b(s, t - 1, t) \varepsilon(s).\tag{1}
\]

In this equation: \( \varepsilon(s) > 0 \) is the longevity innovation or shock in year \( s \) [see Cairns, 2007, equation 5, Olivier and Jeffery, 2004, and Smith, 2005]; while \( b(s, t - 1, t) \) is a normalizing constant, specific to each pair of dates \( s \) and \( t \), and to each cohort, that

\(^2\) Because markets are incomplete, the choice of pricing measure cannot be unique, and must, therefore, be an additional modeling assumption. A risk-neutral pricing measure has the attraction that it ensures that different longevity-dependent securities will be consistently priced. An alternative would be to use a distortion approach as we did in the earlier companion study Dowd et al. [2006], but we now prefer to avoid that because it applies a single market price of risk to all quantities regardless of their uncertainty or timing. The user, our hypothetical pension fund, must therefore make some assumption about what the risk-neutral measure might entail. This said, it can always choose to adopt the real probability measure as its risk-neutral measure, and we might imagine that many pensions would choose to do so.
ensures consistency of prices under the risk-neutral pricing measure. This means that the probability of survival to \( t \) is affected by each of the 1-year longevity shocks \( \varepsilon(1) \), \( \varepsilon(2) \), \( \ldots \), \( \varepsilon(t) \). From that it follows that \( S(t) \), the probability of survival to \( t \), is given by:

\[
S(t) = \prod_{s=1}^{t-1} p(s-1,s-1,s)^{b(s-1,s)_{\varepsilon(s)}} = \prod_{s=1}^{t-1} p(0,s-1,s)\sum_{u=0}^{s-1}b(s-1,s)_{\varepsilon(u)}
\]

(2)

The longevity shocks \( \varepsilon(1) \), \( \varepsilon(2) \), \( \ldots \), \( \varepsilon(t) \) can be modeled by the following transformed beta distribution:

\[
\varepsilon(s) = 2y(s)
\]

(3)

where \( y(s) \) is beta-distributed. Since the beta is defined over the domain \([0,1]\), the transformed beta \( \varepsilon(s) \) is distributed over domain \([0,2]\), where \( \varepsilon(s)<1 \) indicates that longevity improved, and \( \varepsilon(s)>1 \) indicates the opposite.

The model user, our pension fund, is assumed to have a view about future longevity dynamics. The key feature of this view is a set of beliefs about the mean and variance of future mortality rates for individuals of different ages. These beliefs can then be used to

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3 The normalizing constants, \( b(s,t-1,t) \), are known at time \( s-1 \). Under most models, the \( b(s,t-1,t) \) are very close to 1, and for practical purposes these might be dropped.

4 As an alternative to the scaled Beta distribution, Olivier and Jeffery [2004], Smith [2005] and Cairns [2007] use a Gamma distribution with mean 1 under the risk-neutral measure. Under this Gamma model, the normalizing constants \( b(s,t-1,t) \) can be calculated analytically at time \( s-1 \).
calibrate the two parameters of the underlying beta distribution: the longevity shocks \( \varepsilon(1), \varepsilon(2), \ldots, \varepsilon(t) \) will then reflect the user’s view about future longevity.\(^5\)

2.2 The Dowd et al. [2006] pricing methodology

We now explain our pricing methodology in the context of the vanilla survivor swap structure analyzed in Dowd et al. [2006]. This contract is predicated on a benchmark cohort of given initial age. On each of the payment dates, the contract calls for the fixed-rate payer to pay the notional principal multiplied by a fixed proportion \((1 + \pi)H(t)\) to the floating-rate payer and receive in return the notional principal multiplied by \(S(t)\), where \(\pi\) is the swap premium or swap price which is factored into the fixed-rate payment.\(^6\) \(\pi\) is set when the contract is agreed and remains fixed for its duration.

Had the swap been a vanilla interest-rate swap (IRS), we could then have used the spot-rate curve to determine the values of both fixed and floating leg payments. We would have invoked zero-arbitrage to determine the fixed rate that would make the values of both legs equal, and this fixed rate would be the price of the swap. In the present context, however, this is not possible because markets are incomplete, and market incompleteness means that there is no longevity spot-rate curve to use.

\(^{5}\) The user’s view might also incorporate beliefs about the temporal pattern of longevity innovations. For example, the user might believe the innovations \(\varepsilon(t)\) and \(\varepsilon(t-1)\) are positively correlated.

\(^{6}\) Strictly speaking, the contract would call for the exchange of the difference between \((1 + \pi)H(n)\) and \(S(n)\): the fixed rate payer would pay \((1 + \pi)H(n)-S(n)\) if \((1 + \pi)H(n)-S(n)>0\), and the floating rate payer would pay \(S(n)-(1 + \pi)H(n)\) if \((1 + \pi)H(n)-S(n)<0\). We ignore this detail in the text.
To obtain a value of $\pi$, we therefore need a valuation methodology that can be applied in an incomplete-markets setting, and it is for this reason that we assumed a risk-neutral pricing process section 2.1 above. The present value of the floating-leg payments is then the expectation of $S(t)$ taken under the assumed probability measure and can be obtained by stochastic simulation. Under our illustrative model, this is given by:

$$E[S(t)] = E \left[ \prod_{s=0}^{t-1} p(0,s-1,s) \sum_{u=1}^{b(u,s-1,s)} e^{u} \right]$$

with the expectation taken under the assumed risk-neutral measure. The premium $\pi$ is then set so that the swap value is zero at inception. Hence, if $E[S(t)]$ denotes each time-$t$ expected floating-rate payment under the risk-neutral pricing measure, and if $D_t$ denotes the price at time zero of a bond paying $1$ at time $t$, then the fair value for a $k$-period survivor swap requires:

$$(1 + \pi) \sum_{t=1}^{k} D_t H(t) = \sum_{t=1}^{k} D_t E[S(t)]$$

$$\therefore \quad \pi = \frac{\sum_{t=1}^{k} D_t E[S(t)]}{\sum_{t=1}^{k} D_t H(t)} - 1$$

From this structure, it becomes possible to price a range of related derivatives securities.
2.3 Generalizing the Dowd et al. [2006] pricing methodology

The pricing model set out above can be generalized to a wide range of related derivatives. For ease of presentation, assume that all members of the cohort – whose survivorship is the underlying asset in these derivatives – are of the same age and that payments due under the derivatives are made annually. We denote this age during the evolution of the derivatives by the following subscripts:

- $t$: their age at the time of the contract agreement
- $s$: their age at the time of the first payment
- $f$: their age at the time of the final payment
- $n$: their age at the time of any given anniversary ($t \leq n \leq f$)

Let us also denote

- $N$: the size of the cohort at age $t$
- $D_n$: the discount factor from age $t$ to age $n$
- $Y_n$: the payment per survivor due at age $n$ ($= 0$ for $n < s$)

Now note that the present value (at time $t$) of a fixed payment due at time $n$ is:

$$ (1 + \pi)N Y_n D_n H(n) $$

(7)

From this, it follows that the present value, at time $t$, of the payments contracted by the pay-fixed party to the swap is
\[(1 + \pi)N \sum_{n=t+1}^{f} Y_n D_n H(n) \tag{8}\]

which – conditional on \( \pi \) – can be determined easily at time \( t \) from the spot-rate curve.

Following the same approach as with the vanilla survivor swap, the present value of the floating rate leg is

\[N \times E \left[ \sum_{n=t+1}^{f} Y_n D_n S(n) \right] = N \sum_{n=t+1}^{f} Y_n D_n E[S(n)] \tag{9}\]

Since a swap has zero value at inception, we then combine (8) and (9) to calculate a premium, \( \pi_{s,f} \), for any swap-type contract, valued at time \( t \), whose payments start at age \( s \) and finish at age \( f \). This premium is given by:

\[\pi_{s,f} = \frac{\sum_{n=t+1}^{f} Y_n D_n E[S(n)]}{\sum_{n=t+1}^{f} Y_n D_n H(n)} - 1 \tag{10}\]

3. PRICING OTHER SURVIVOR DERIVATIVES
We now use the pricing methodology outlined in the previous section to price some key survivor derivatives.

3.1 Survivor forwards

Just as an interest-rate swap is essentially a portfolio of FRA contracts, so a survivor swap can be decomposed into a portfolio of survivor forward contracts. Consider two parties, each seeking to fix payments on the same cohort of 65-year-old annuitants. The first enters into a $k$-year, annual-payment, pay-fixed swap as described above, and with premium, $\pi$. The second enters into a portfolio of $k$ annual survivor forward contracts, each of which requires payment of the notional principal multiplied by $(1 + \pi_n)H(n)$ and the receipt of the notional principal multiplied by $S(n)$, $n = 1, 2, \ldots k$. Note that in this second case, $\pi_n$ differs for each $n$. Since the present value of the commitments faced by the two investors must be equal at the outset, it must be that:

\[
(1 + \pi) \sum_{n=1}^{k} D_s H(n) = \sum_{n=1}^{k} D_s H(n)(1 + \pi_n)
\]

(11)

\[
\therefore \pi = \frac{\sum_{n=1}^{k} D_s H(n)\pi_n}{\sum_{n=1}^{k} D_s H(n)}
\]

(12)
Hence it follows that $\pi$ in the survivor swap must be equal to the weighted average of the individual values of $\pi_n$ in the portfolio of forward contracts, in the same way that the fixed rate in an interest-rate swap is equal to the weighted average of the forward rates.

### 3.2 Forward survivor swaps

Given the existence of the individual values of $\pi_n$ in the portfolio of forward contracts, it becomes possible to price forward survivor swaps. In such a contract, the parties would agree at time zero, the terms of a survivor swap contract which would commence at some specified time in the future. Not only would such a contract meet the needs of those who are committed to providing pensions in the future, but this instrument could also serve as the hedging vehicle for survivor swaptions – see below.

The pricing of such a contract would be quite straightforward. As shown above, the position could be replicated by entering into an appropriate portfolio of forward contracts. Thus $\pi_{\text{forward swap}}$ – the risk premium for the forward swap contract – must equal the weighted average of the individual values of $\pi_n$ used in the replication strategy. $\pi_{\text{forward swap}}$ can then be derived directly from equation (10).

### 3.3 Basis swaps

Dowd et al. [2006] also discuss, but do not price, a floating-for-floating swap, in which the two counterparties exchange payments based on the actual survivorships of two different cohorts. Following practice in the interest-rate swaps market, such contracts
should be called basis swaps. The analysis above shows how such contracts could be priced. First, consider two parties wishing to exchange the notional principal\(^7\) multiplied by the actual survivorship of cohorts \(j\) and \(k\). Assume equal notional principals and denote the risk premiums and expected survival rates for such cohorts by \(\pi_i\) and \(\pi_k\) and \(H_j(n)\) and \(H_k(n)\), respectively. Given the existence of vanilla swap contracts on each cohort, the present values of the fixed leg of each such contract will be 

\[
(1 + \pi_j) \sum_{n=1}^{T} D_n H_j(n) \quad \text{and} \quad (1 + \pi_k) \sum_{n=1}^{T} D_n H_k(n),
\]

respectively, and the zero value argument shows that these must also be the present values of the expected floating rate legs. It is then possible to calculate with certainty, an exchange factor, \(\kappa\), such that

\[
(1 + \pi_j) \sum_{n=1}^{T} D_n H_j(n) = \kappa (1 + \pi_k) \sum_{n=1}^{T} D_n H_k(n)
\]

and, hence

\[
\kappa = \frac{(1 + \pi_j) \sum_{n=1}^{T} D_n H_j(n)}{(1 + \pi_k) \sum_{n=1}^{T} D_n H_k(n)}
\]

from which it follows that the fair value in a floating-for-floating basis swap requires one party to make payments determined by the notional principal multiplied by \(S_j(n)\) and the other party to make payments determined by the notional principal multiplied by \(\kappa S_k(n)\);

\(^7\) Following practice in the interest rate swaps market, we avoid constant reference to the notional principal henceforth by quoting swap prices as percentages. The notional principal in survivor swaps can be expressed as the cohort size, \(N\), multiplied by the payment per survivor at time \(n, Y_n\).
κ is determined at the outset of the basis swap and remains fixed for the duration of the contract.

The same approach can be used to price forward basis swaps, in which case, following earlier analysis, κ is given by:

\[ \kappa = \frac{(1 + \pi_j) \sum_{n=s}^{f} D_n H_j(n)}{(1 + \pi_k) \sum_{n=s}^{f} D_n H_k(n)} \]  

(15)

3.4 Cross-currency basis swaps

We turn now to price a cross-currency basis swap, in which the cohort-\( j \) payments are made in one currency and the cohort-\( k \) payments in another. The single currency floating-for-floating basis swap analyzed in the preceding sub-section required the cohort-\( j \) payer to pay \( S_j(n) \) at each payment date and to receive \( \kappa S_k(n) \). Now consider a similar contract in which the cohort-\( j \) payments are made in currency \( j \) and the cohort-\( k \) payments made in currency \( k \). Assume the spot exchange rate between the two currencies is \( F \) units of currency \( k \) for each unit of currency \( j \). 8

From the arguments above, we can determine the present value of each stream –

\[ (1 + \pi_j) \sum_{n=1}^{f} D_n H_j(n) \]  and \[ (1 + \pi_k) \sum_{n=1}^{f} D_n H_k(n) \], respectively – each expressed in their

8 In foreign exchange markets parlance, currency \( j \) is the base currency and currency \( k \) is the pricing currency.
respective currencies. Multiplying the latter by $F$ then expresses the value of the cohort-$k$ stream in units of currency $j$. The standard requirement that the two streams have the same value at the time of contract agreement is again achieved by determining an exchange factor, $\kappa_{FX}$. In the present case, this exchange factor, $\kappa_{FX}$ is given by:

\[
(1 + \pi_j) \sum_{n=1}^{f} D_n H_j(n) = \kappa_{FX} F (1 + \pi_k) \sum_{n=1}^{f} D_n H_k(n) \tag{16}
\]

\[
\therefore \kappa_{FX} = \frac{(1 + \pi_j) \sum_{n=1}^{f} D_n H_j(n)}{F (1 + \pi_k) \sum_{n=1}^{f} D_n H_k(n)} \tag{17}
\]

Thus, in the case of a floating-for-floating cross-currency basis swap, on each payment date, $n$, one party will make a payment of the notional principal multiplied by $S_j(n)$ and receive in return a payment of $\kappa_{FX} S_k(n)$. Each payment will be made in its own currency, so that exchange rate risk is present. However, in contrast with a cross-currency interest-rate swap, there is no exchange of principal at the termination of the contract, so the exchange rate risk is mitigated.

The same procedure is used for a forward cross-currency basis swap, except that the summation in equations (16) and (17) above is from $n = s$ to $f$, rather than from $n = 1$ to $f$. 


Since the desire is to equate present values, it should be noted that the spot exchange rate, \( F \), is applied in this equation, rather than the forward exchange rate.\(^9\)

### 3.5 Futures contracts

The wish to customize the specification of the cohort(s) in the derivative contracts described above implies trading in the over-the-counter (OTC) market. However, an exchange-traded instrument offers attractions to many. As shown above, the uncertainty in survivor swaps is captured in the factor \( \pi \), and a futures contract with \( \pi \) as the underlying asset would serve a useful function both as a hedging vehicle and for investors who wished to achieve exchange-traded exposure to survivor risk, in much the same way as the Eurodollar futures contract is based on 3-month Eurodollar LIBOR.

Thus, if the notional principal were $1 million and the timeframe were 1 year, a long position in a December futures contract at a price of \( \pi = 3\% \) would notionally commit the holder to pay $1.03 million multiplied by the expected size of the cohort surviving and to receive $1 million multiplied by the actual size. This is a notional commitment only: in practice, the contracts would be cash-settled, so that if the spot value of \( \pi \) at the December expiry, which we denote \( \pi_{\text{expiry}} \), were 4\%, the investor would receive a cash payment of $10,000: \textit{i.e.} (4–3)\% of $1 million.\(^{10}\)

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\(^9\) The foreign exchange risk could be eliminated by use of a survivor swap contract in which the payments in one currency are translated into the second currency at a predetermined exchange rate, similar to the mechanics of a quanto option. Derivation of the pricing of such a contract is left for future research.

\(^{10}\) Recall, however, that \( \pi \) can take values between \(-1\) and \(1\). Since negative values are rare for traded assets, this raises the issue of whether user systems are able to cope. To avoid such problems, the market for \( \pi \) futures contracts could either be quoted as \((1 + \pi)\), with \( \pi \) as a decimal figure, or else follow interest-rate futures practice and be quoted as \((100 - \pi)\) with \( \pi \) expressed in percentage points.
The precise cohort specification would need to be determined by research among likely users of the contracts. Too many cohorts would spread the liquidity too thinly across the contracts: too few cohorts would lead to excessive basis risk.

Determination of the settlement price at expiry might be achieved by dealer poll. Such futures contracts could be expected to serve as the principal driver of price discovery in the Life Market, with dealers in the OTC market using the futures prices to inform their pricing of customized survivor swap contracts.

4. SURVIVOR SWAPTIONS

Where there is demand for linear payoff derivatives, such as swaps, forwards, and futures, there is generally also demand for option products. An obvious example is a survivor swaptions contract.

4.1 Specification of swaptions

The specification of such options is quite straightforward. Consider a forward survivor swap, described above, with premium $\pi_{\text{forwardswap}}$. A swaption would give the holder the right but not the obligation to enter into a swap on specified terms. Clearly, the exercise decision would depend on whether the market rate of the $\pi$ at expiry for such a swap was greater or less than $\pi_{\text{forwardswap}}$. Thus, in the case described, $\pi_{\text{forwardswap}}$ is the strike price of the swaption. Of course, the strike price of the option does not have to be $\pi_{\text{forwardswap}}$, 
but can be any value that the parties agree. However, using \( \pi_{\text{forwardswap}} \) as an example shows how put-call\(^{11}\) parity applies to such swaptions. An investor who purchases a payer swaption, at strike price \( \pi_{\text{forwardswap}} \), and writes a receiver swaption with an identical specification has synthesized a forward survivor swap. Since such a contract could be opened at zero cost, it follows that a synthetic replication must also be available at zero cost. Hence the premium paid for the payer swaption must equal the premium received for the receiver swaption.

The exercise of these swaptions could be settled either by delivery (i.e. the parties enter into opposite positions in the underlying swap) or by cash, in which case the writer pays the holder \( \max\left[0, \phi \pi_{\text{expiry}} - \phi \pi_{\text{strike}}\right] N \sum_{n=1}^{f} D_{\text{expiry},n} Y_n H(n) \) with \( \phi \) set as +1 for payer swaptions and -1 for receiver swaptions, \( \pi_{\text{expiry}} \) representing the market value of \( \pi \) at the time of swaption expiry, \( \pi_{\text{strike}} \) representing the strike price of the swaption and \( D_{\text{expiry},n} \) representing the price at option expiry of a bond paying 1 at time \( n \).\(^{12}\)

### 4.2 Pricing swaptions

\(^{11}\) In swaptions markets, usage of terms such as put and call can be confusing. Naming such options payer (i.e. the right to enter into a pay-fixed swap) and receiver (i.e. the right to enter into a receive-fixed swap) swaptions is preferable. We denote the options premia for such products as \( P_{\text{payer}} \) and \( P_{\text{receiver}} \) respectively.

\(^{12}\) Under Black-Scholes (1973) assumptions, interest rates are constant, so that \( N \sum_{n=1}^{f} D_{\text{expiry},n} Y_n H(n) \) is known from the outset. Let us call this the settlement sum. Following the approach in footnote 8, we can dispense with constant repetition of the settlement sum by expressing option values in percentages and recognising that these can be turned into a monetary amount by multiplying by the settlement sum.
Our survivor swaptions are specified on the swap premium $\pi$ as the underlying, and this raises the issue of how $\pi$ is distributed. In a companion paper, Dawson et al. [2009] suggest that $\pi$ should be (at least approximately) normal, and report Monte Carlo results that support this claim.\textsuperscript{13} We can therefore state that $\pi$ is approximately $N(\pi_{\text{forwardswap}}, \sigma^2)$, where $\sigma^2$ is expressed in annual terms in accordance with convention. Normally distributed asset prices are rare in financial assets, because such a distribution permits the asset price to become negative. In the case of $\pi_{\text{forwardswap}}$, however, negative values are perfectly feasible.

Dawson et al [2009] derive and test a model for pricing options on assets with normally distributed prices and application of their model to survivor swaptions gives the following formulae for the swaption prices:

\begin{align*}
P_{\text{payer}} &= e^{-r\tau} \left( \left( \pi_{\text{forwardswap}} - \pi_{\text{strike}} \right) N(d) + \sigma \sqrt{\tau} N'(d) \right) \quad (18) \\
P_{\text{receiver}} &= e^{-r\tau} \left( \left( \pi_{\text{strike}} - \pi_{\text{forwardswap}} \right) N(-d) + \sigma \sqrt{\tau} N'(d) \right) \quad (19) \\
d &= \frac{\pi_{\text{forwardswap}} - \pi_{\text{strike}}}{\sigma \sqrt{\tau}} \quad (20)
\end{align*}

\textsuperscript{13} More precisely, the large Monte Carlo simulations (250,000 trials) across a sample of different sets of input parameters reported in Dawson et al. [2009] suggest that $\pi_{\text{forwardswap}}$ is close to normal but also reveal small but statistically significant non-zero skewness values. Furthermore, whilst excess kurtosis is insignificantly different from zero when drawing from beta distributions with relatively low standard deviations, the distribution of $\pi_{\text{forwardswap}}$ is observed to become increasingly platykurtic as the standard deviation of the beta distribution is increased. Our option pricing model can deal with these effects in the same way as the Black-Scholes models deal with skewness and leptokurtosis. In this case of platykurtosis, a volatility frown, rather than a smile, is dictated.
In (18) – (20) above, $r$ represents the interest rate, $\tau$ the time to option maturity and $\sigma$ the annual volatility of $\pi_{\text{forwardswap}}$. Apart from replacing geometric Brownian motion with arithmetic Brownian motion, this valuation model is predicated on the standard Black-Scholes [1973] assumptions, including, *inter alia*, continuous trading in the underlying asset. Naturally, we recognize that, at present, no such market exists, but we argue that this as a reasonable assumption since survivor swaptions cannot exist without survivor swaps.

The use of this model in practice would, therefore, almost inevitably involve some degree of basis risk. This arises, in part, because it is unlikely that a fully liquid market will ever be found in the specific forward swap underlying any given swaption. A liquid market in the $\pi$ futures described above would mitigate these problems, however.\textsuperscript{14} Furthermore, survivor swaption dealers will likely need to hedge positions in swaptions on different cohorts, which will be self-hedging to a certain extent and so reduce basis risk. We could also envisage option portfolio software that would translate some of the remaining residual risk into futures contract equivalents, thus dictating (and possibly automatically submitting) the orders necessary for maintaining delta-neutrality.

Most liquid futures markets create a demand for futures options, and this leads to the possibility of $\pi$ futures options. Pricing such contracts is also accomplished in (18)-(19)

\textsuperscript{14} Given a variety of cohorts, basis risk could be a problem, but as noted earlier, an important pre-condition of futures introduction is research among industry participants to optimize the number of cohorts for which $\pi$ futures would be introduced. A large number of cohorts decreases the potential basis risk, but spreads the liquidity more thinly across the contracts.
above. All that is necessary is to substitute $\pi_{\text{futures}}$ for $\pi_{\text{forwardswap}}$ as the value of the optioned asset.

4.3 The Greeks

Dawson et al [2009] derive the Greeks for their option model with a normally distributed underlying. Table 1 below consolidates the formulae for option values and the delta, gamma, rho, theta and vega for payer and receiver swaptions.

*Insert Table 1 about here*

4.4 Survivor caps and floors

The parallels with the interest rate swaps market can be carried still further. In the interest rate derivatives market, caps and floors are traded, as well as swaptions. These offer more versatility than swaptions, since each individual payment is optioned with a caplet or a floorlet, whilst a swaption, if exercised, determines a single fixed rate for all payments. The extra optionality comes at the expense of a significantly increased option premium however. Similar caplets and floorlets can be envisaged in the market for survivor derivatives, and can be priced using (18) and (19) with the $\pi$ value for the survivor forward contract serving as the underlying in place of $\pi_{\text{forwardswap}}$.

5. HEDGING APPLICATIONS
In this section, we consider applications of the securities presented above. By way of example, we consider a pension fund with a liability to pay $10,000 annually to each survivor of a cohort of 1,000 65-year-old males. Using the same life tables as Dowd et al. [2006], the present value of this liability is approximately $106.1 million and the pension fund is exposed to considerable survivor risk. We consider several strategies to mitigate this risk. In pricing the various securities applied, we use the models presented earlier in this paper and, for the sake of example, we assume a flat yield curve at 6%, values of 5,050 and 4,950 for the beta distribution used to model $\varepsilon_i$ in (3) above. The beta values determine the volatility of the hedging instrument used in survivor swaption valuation, and this case the volatility is 0.485%.

The first hedging strategy which the fund might undertake is to enter into a 50-year survivor swap. Using the parameters from the previous paragraph gives a swap rate of 3.1667%. Entering a pay-fixed swap at this price would remove the survivor risk entirely from the pension fund, but increase the present value of its liabilities to approximately $109.5 million (i.e. $106.1 million $\times 1.031667$).

The second hedging strategy has the pension fund choosing to accept survivor risk for the next five years, and entering into a forward swap today to hedge survivor risk from age 71 onwards. The value of $\pi$ is now 5.0910%. However, since this premium will not be paid for the first five years of the liability, when by definition, survivorship will be highest, the impact of the higher premium is much reduced. To see this, note that the $106 million referred to above can then be decomposed into two components: the present
value of the liabilities for the first five years, worth approximately $40.7 million, and the present value for the remaining years, worth approximately $65.4 million. Thus, accepting survivor risk for the first five years, but hedging it for the remainder of the survivorship would increase the present value of the fund’s liabilities to approximately $109.4 million ($40.7 million + $65.4 million \times 1.05091).

The fact that both strategies considered so far appear to increase the present value of the fund’s liabilities to approximately $109.5 million implies that there is almost zero benefit, and therefore almost zero cost, to hedging for the first five years. This is borne out by analysis of the forward curve. We note first that an alternative to the 50-year swap is for the hedger to use a portfolio of 50 survivor forward contracts. The no-arbitrage arguments presented earlier then mean that the cost of hedging in this manner must equal the cost of hedging with the swap. In this case, this cost amounts to about $3.4 million ($106.1 \times 0.031667). Figure 1 below shows the distribution of this cost across the evolving ages of the cohort. It can be seen the cost is minimal during the early years\(^{15}\) (since there has been little time for longevity shocks to make any impact) and during the late years (when the cohort size has declined to very small numbers). Instead, the maximum cost occurs during the middle years, when there has been sufficient time for longevity shocks to make a significant impact and the cohort is still sufficiently large for this impact to be economically significant.

\(^{15}\) The \(\pi\) value for a swap covering the just first five years of this exposure is a mere 0.0798%, representing a hedging cost of approximately $32,500.
These three hedging strategies fix the commitments of the pension fund, either for the entire period of survivorship or for all but the first five years, and this means that the fund would have no exposure to any financial benefits from decreasing survivorship. These benefits could, however, be obtained by our fourth hedging strategy, namely a position in a five-year payer swaption. Again, the fund would accept survivor risk for the first five years, but would then have the right, but not the obligation, to enter a pay-fixed swap on pre-agreed terms. Using the same beta parameters as above, the annual volatility of the forward contract is 0.51% and the premium for an at-the-money forward swaption is 0.3378%. It was seen above that the present value of these liabilities at age 65 was $65.4 million. Applying our swaption formula, the pension fund would pay an option premium of approximately $221,000 to gain the right, but not the obligation to fix the payments at a π rate of 5.0910% thereafter. If survivorship declines, the fund will not exercise the option and will have lost merely the $221,000 swaption premium, but then reaps the benefit of declining survivorship.

Rather more optionality could be obtained through a survivor cap, rather than a survivor swaption. As described above, this is constructed as a portfolio of options on survivor forwards. The expiry value of each option is \( NYH(n)\text{Max}[0, \pi_{\text{settlement}} - \pi_{\text{strike}}] \), in which \( \pi_{\text{settlement}} \) refers to the value of \( \pi \) prevailing for that particular payment at the time the settlement is due. Its value can be ascertained from (11) which simplifies in this case to
\((H(n) - S(n))/S(n)\).\(^{16}\) Thus, on each payment date, \(n\), the pension fund holding a survivor cap effectively pays \(NY(1+\text{Min}[\pi_{\text{settlement}}, \pi_{\text{strike}}])H(n)\) on each payment date \(n\) and receives \(NYS(n)\) in return, with this receipt designed to match their liability to their pensioners. This extra optionality comes at a price: it was noted above that the option premium for a five-year survivor swaption, at strike price 5.0910%, with our standard parameters, was approximately $220,000. The equivalent survivor cap (in which the pension fund again accepts survivor risk for the first five years, but hedges it with a survivor cap again struck at 5.0910% for the remaining 45 years) would carry a premium of $1.84 million.

Our next hedging strategy is a zero-premium collar. Again using the same beta parameters, the premium for a payer swaption with a strike price of 5.5% is 0.2068%. The same premium applies to a receiver swaption with a strike price of 4.682%. Thus a zero-premium collar can be constructed with a long position in the payer swaption financed by a short position in the receiver swaption. With such a position, the pension fund would be committed to a pay-fixed swap contract at cohort age 70, but at a rate of no more than 5.5% and no less than 4.682%.

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\(^{16}\) Equation (11) is \((1 + \pi) \sum_{n=1}^{k} D_{\pi} H(n) = \sum_{n=1}^{k} D_{\pi} [S(n)]\). However, since the forward contract refers to just one specific maturity, \(n\), the summation operator is redundant. Furthermore, since there is no time difference between this valuation and the payment, the discount factors are irrelevant. Finally, at expiry, \(S(n)\) is known with certainty, so that the expectations operator is irrelevant. Thus \((1 + \pi)H(n) = S(n)\) and so \(\pi = (H(n) - S(n))/H(n)\).
One downside to such a zero-premium collar is that the pension fund puts a floor on its potential gains from falling survivorship. If it wishes to have a zero-premium collar and retain upside potential from falling survivorship, an alternative is to finance the purchase of the payer swaption by the sale of a receiver swaption with the same strike price. Since the payer swaption is out of the money, its premium is considerably less than that of the in-the-money receiver swaption – 0.2068% versus 0.5124%. Thus to finance a payer swaption on the $65.4 million liabilities, it would be necessary to sell a receiver swaption on only $26.4 million (= $65.4 million × 0.002068 ÷ 0.005124) of liabilities. The pension fund would then enjoy, at zero premium, complete protection against the survivor premium rising above 5.5%, and unlimited participation, albeit at a little less than 60c on the dollar, if the survivor premium turns out to be less than this.

The hedging strategies presented so far in this section serve to transfer the survivor risk embedded in a pension fund to an outside party. In all cases, this is done either at a cost: either an explicit financial cost or, in the case of the zero-premium option structures, at the willingness to forego some of the financial benefits of falling survivorship, i.e., at an opportunity cost. A quite different alternative that avoids these costs is simply for the pension fund to diversify its exposure. Using a basis swap or a cross-currency basis swap, the pension fund could swap some of its exposure to the existing cohort for an exposure to a different cohort (either in its domestic economy or overseas). Hence, in return for receiving cashflows to match some of its obligations to its own pensioners, it would assume liability for paying according to the actual survivorship of a different cohort. As the derivations of equations (15) and (17) show, this does not change the value of the
pension fund’s liabilities, but, assuming less than perfect correlation between the survival rates of the two cohorts, enables the pension fund to enjoy the benefits of diversification.

It would be wrong to leave this discussion of available hedging techniques without a brief discussion of structured products. The financial press has already begun to make comparisons between survivor derivatives and credit derivatives. The creative structuring, which has been so prevalent in credit derivatives, can be expected in survivor derivatives as well. The simplest example would be for the pension fund to issue a 50-year survivor-linked note, paying a coupon of LIBOR + 3.1667% (i.e., the premium for a 50-year survivor swap). The holder of the note would enjoy a significant premium over a standard floating rate note, but the cashflows paid would be adjusted each year according to the evolution of the cohort survivorship. Thus, the pension fund would be transferring the survivor risk, but in a funded, rather than an unfunded form.

The next step up in complexity would be for a market for tranched products on a defined cohort. The notional principal on the instruments would be linked to the evolving survivorship and the impact of mortality would first be borne by the equity tranche, until that is eliminated at which point the impact would be borne by successively higher tranches.

Needless to say, many other types of structured note and tranched products based on survivor underlyings can also be envisaged.

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17 See, for example, Wighton and Tett [2006].
6. CONCLUSION

The products presented and priced in this paper go a long way towards completing the market for survivor derivatives. Further developments can be expected. Quanto features could be incorporated in cross-currency products to eliminate currency risk. Knock-out and knock-in features might also be anticipated. For example, a pension fund might be quite willing to forego protection against increasing survivorship in the event of an avian flu pandemic and would buy a payer swaption which knocks out if mortality rises above a predetermined threshold. Such a payer swaption would specify a low value of $\pi$ as the knock-out threshold. Alternatively, such a fund might seek protection contingent on a major breakthrough in the treatment of cancer and would thus buy a payer swaption which knocks in if survivorship rises above a predetermined threshold. Such a payer swaption would specify a high value of $\pi$ as the knock-in threshold.

Survivorship is a risk of considerable importance to developed economies. It is surprising that the market has been so slow to develop derivative products to manage such risk. However, parallels with the interest rate and credit derivatives markets seem apposite: once the initial products were launched, the growth in these markets was rapid. Survivor derivatives are now beginning to emerge and manage a risk which is arguably more serious than that managed by credit derivatives. Market completion is both important and inevitable.
REFERENCES


See http://www.actuaries.ie/Resources/events papers/PastCalendarListing.htm


*Financial Times*, November 22.
Figure 1: Hedging cost using individual forward contracts
Table 1 – Summary of the option pricing model and its Greeks

<table>
<thead>
<tr>
<th></th>
<th>Payer swaptions</th>
<th>Receiver swaptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option value</td>
<td>$e^{-rt} \left[ (\pi_{\text{forward swap}} - \pi_{\text{strike}}) N(d) - \sigma \sqrt{\tau} N'(d) \right]$</td>
<td>$e^{-rt} \left[ (\pi_{\text{strike}} - \pi_{\text{forward swap}}) N(-d) + \sigma \sqrt{\tau} N'(d) \right]$</td>
</tr>
<tr>
<td>Delta</td>
<td>$e^{-rt} N(d)$</td>
<td>$-e^{-rt} N(-d)$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{e^{-rt}}{\sigma \sqrt{\tau}} N'(d)$</td>
<td>$\frac{e^{-rt}}{\sigma \sqrt{\tau}} N'(d)$</td>
</tr>
<tr>
<td>Rho (per percentage point rise in rates)</td>
<td>$\frac{-\tau P_{\text{payer}}}{100}$</td>
<td>$\frac{-\tau P_{\text{receiver}}}{100}$</td>
</tr>
<tr>
<td>Theta (for 1 day passage of time)</td>
<td>$2 \sqrt{\tau} r P_{\text{payer}} - e^{-rt} \sigma N'(d)$ \bigg/ 730\sqrt{\tau}$</td>
<td>$2 \sqrt{\tau} r P_{\text{receiver}} - e^{-rt} \sigma N'(d) + 4 \sqrt{\tau} r e^{-rt} \left( \pi_{\text{forward swap}} - \pi_{\text{strike}} \right)$ \bigg/ 730\sqrt{\tau}$</td>
</tr>
<tr>
<td>Vega (per percentage point rise in volatility)</td>
<td>$\frac{e^{-rt} \sqrt{\tau} N'(d)}{100}$</td>
<td>$\frac{e^{-rt} \sqrt{\tau} N'(d)}{100}$</td>
</tr>
</tbody>
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