Localized Realized Volatility Modeling

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Volatility

Risk management, derivative pricing and hedging, portfolio selection, market making, etc. require accurate volatility estimates and forecasts, see Engle & Patton (2001).

High-frequency information $\rightarrow$ Realized Volatility (RV):

- a consistent estimator of integrated volatility
- shows nice forecast properties

see Andersen, Bollerslev, Diebold and Labys (2001).
An important realized volatility fact

Figure 1: Sample autocorrelations of log. RV for different sample periods.
A dual view on the long memory diagnosis

- **The long memory point of view:**
  Volatility is generated by long memory processes, i.e. fractionally integrated, $I(d)$, processes.

- **The short memory point of view:**
  Volatility may equally well be generated by a short memory process with structural breaks, see e.g. Diebold and Inoue (2001), Granger and Hyung (2004).
  Example: GARCH model with changing parameters, see e.g. Mikosch and Stärică (2004), Čižek, Härdle and Spokoiny (2009).
Objectives

- Develop a localized modeling for RV
  - For a fixed time point, find a past time interval for which a local volatility model is a good approximator.
  - The localized model is then used to predict RV.
  - The time interval is determined by modern nonparametric statistical methods.

- Investigate the dual view on volatility phenomenon
Outline

1. Motivation ✓
2. Realized volatility
3. Localized realized volatility approach
4. Long memory models
5. Empirical analysis
6. Conclusion
Realized volatility

\[ \widetilde{RV}_t = \sum_{j=1}^{M} r_{t,j}^2, \]

with \( r_{t,j} = p_{t,n_j} - p_{t,n_{j-1}}, j = 1, \ldots, M, \) and \( p_{t,n_j} \) the logarithmic price observed at time point \( n_j \) of trading day \( t \).

It converges to the quadratic variation for \( M \to \infty \), see Andersen and Bollerslev (1998), Barndorff-Nielsen and Shephard (2002b).

- Example: for exchange spot rates, \( M \approx 275,000 \) (Reuters)
- High-frequency data is subject to microstructure noise, see Martin et al. (2007)
Realized volatility based on kernel estimators

Tukey-Hanning kernel:

\[
RV_t = \widehat{RV}_{t,1} + \sum_{h=1}^{H^*} k \left( \frac{h - 1}{H^*} \right) (\gamma_{t,h} + \gamma_{t,-h})
\]

where

\[
k(x) = \sin^2 \left\{ \frac{\pi}{2} (1 - x)^2 \right\}, \text{ the best option in terms of efficiency}
\]

\[
\gamma_{t,h} = \sum_{j=1}^{M} r_{t,j} r_{t,j-h} \text{ is based on one-minute returns,}
\]

\[
H^* = 5.74 \frac{\overline{RV}_{t,15}}{\sqrt{M}} \text{ with } RV_{t,i} \text{ the realized variance estimator based on } i \text{ minute returns.}
\]

see Barndorff-Nielsen, Hansen, Lunde and Shephard (2008)
Data

Our empirical analysis is based on S&P500 index futures from January 2, 1985 to February 4, 2005.

<table>
<thead>
<tr>
<th>Series</th>
<th>Mean</th>
<th>Std.Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>LB(21)(^{(1)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>RV(_t)</td>
<td>1.09</td>
<td>8.70</td>
<td>55.59</td>
<td>3412</td>
<td>1204</td>
</tr>
<tr>
<td>log(RV(_t))</td>
<td>-0.53</td>
<td>0.89</td>
<td>0.53</td>
<td>4.99</td>
<td>4.69</td>
</tr>
</tbody>
</table>

\(^{(1)}\) The critical value of this Ljung-Box test is 32.671.

Table 1: Descriptive statistics.
Figure 2: Kernel density estimates (solid line: log. $RV$, shaded area: point-wise 95% confidence intervals, dashed line: normal distribution).
Figure 3: Time evolvement of logarithmic realized volatility of the S&P500 index futures.
Localized realized volatility

LAR(p) model with parameter set \( \theta_t = (\theta_{0t}, \theta_{1t}, \ldots, \theta_{pt}, \sigma_t)^\top \):

\[
\log RV_t = \theta_{0t} + \sum_{i=1}^{p} \theta_{it} \log RV_{t-i} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_t^2),
\]

(1)

Suppose \( \theta_t \equiv \theta^* \) for \( t \in I = [1, T + h] \)

\[
\tilde{\theta}_\tau = \arg\max_{\theta \in \Theta} L(\log RV; I_\tau, \theta) = \arg\max_{\theta \in \Theta} \left\{ -\frac{s}{2} \log 2\pi - s \log \sigma \right. \\
- \frac{1}{2\sigma^2} \sum_{t=\tau-s}^{\tau-1} (\log RV_{t+h} - \theta_0 - \sum_{i=1}^{p} \theta_i \log RV_{t-i})^2 \}
\]

Goal: identify a local homogeneous interval \( I_\tau \) for time point \( \tau \).
Identify local homogeneity

At time point $\tau$, choose a local homogeneous interval from

$$\{I^k_{\tau}\}_{k=1}^K = \{I^1_{\tau}, I^2_{\tau}, \ldots, I^K_{\tau}\}$$

where $I^k_{\tau} = [\tau - s_k, \tau)$ with $0 < s_k < \tau$, which leads to the best possible accuracy of estimation.

- Under local homogeneity $\theta_{\tau-s_k} = \cdots = \theta_{\tau} \equiv \theta^*$ within $I^k_{\tau} = [\tau - s_k, \tau)$:
  
  $\tilde{\theta}^{(k)}_{\tau}$ estimates $\theta^*$ at rate $1/\sqrt{s_k}$

- The modeling bias of approximating LAR(h) increases with $s_k$.

The optimal choice $\hat{I}_{\tau}$: balances the bias and variation.

LRV
Localized realized volatility

\section*{Estimation under local homogeneity}

Given $I_\tau = [\tau - s, \tau)$, the local MLE is:

$$\hat{\theta}_\tau = \arg\max_{\theta \in \Theta} L(\log RV; I_\tau, \theta)$$

Under \textbf{local homogeneity}: $\theta_\tau \equiv \theta^*$, the fitted likelihood ratio measures the estimation risk:

$$LR(I_\tau, \hat{\theta}_\tau, \theta^*) = L(I_\tau, \hat{\theta}_\tau) - L(I_\tau, \theta^*).$$

(2)
Estimation under local homogeneity

The estimation risk $LR(I_\tau, \tilde{\theta}_\tau, \theta^*)$ is stochastically bounded:

$$E_{\theta^*} |LR(I_\tau, \tilde{\theta}_\tau, \theta^*)|^r \leq \xi_r.$$ 

It leads to the confidence set:

$$\mathcal{E}(\varepsilon) = \{\theta : LR(I_\tau, \tilde{\theta}_\tau, \theta^*) \leq \varepsilon\}$$

in the sense that $P_{\theta^*} \{\mathcal{E}(\varepsilon) \not\ni \theta^*\} \leq \alpha$, Polzehl and Spokoiny (2006).
Localized AR(p) model

Interval set \( \{ I^k \} \) for \( k = 1, \cdots, K \):

\[
I^1_T = [\tau - 1w, \tau) \quad I^2_T = [\tau - 1m, \tau) \quad \cdots \quad I^K_T = [\tau - 5y, \tau)
\]

\[
\tilde{\theta}_T^{(1)} \quad \tilde{\theta}_T^{(2)} \quad \cdots \quad \tilde{\theta}_T^{(K)}
\]

- The interval is growing in length.
- Local homogeneity is assumed at \( I^1_T \).
- Final estimate \( \hat{\theta}_T \) is based on a sequential testing.
Localized realized volatility

Sequential testing

Suppose that $I_{k-1}^\tau$ is a homogeneous interval: $\hat{\theta}_T^{(k-1)} = \tilde{\theta}_T^{(k-1)}$. The null hypothesis at step $k$:

$$H_0 : I_T^k \text{ is an homogeneous interval.}$$

Test homogeneity of $I_T^k$: $\hat{\theta}_T^k = \tilde{\theta}_T^k$ or terminates at $I_T^{k-1}$

Test: $\left| LR(I_T^k, \tilde{\theta}_T^k, \hat{\theta}_T^{k-1}) \right|^r \leq \zeta_k$, where $\zeta_k$ is critical value (CV).
Adaptive procedure

1. Initialization: \( \hat{\theta}_1^1 = \tilde{\theta}_1^1 \).

2. \( k = 1 \)
   
   while \(|LR(I_T, \tilde{\theta}_T^{k+1}, \hat{\theta}_T^k)|^r \leq \zeta_{k+1} \) and \( k < K \),
   
   \[
   \begin{align*}
   k & = k + 1 \\
   \hat{\theta}_T^k & = \tilde{\theta}_T^k
   \end{align*}
   \]

3. Final estimate: \( \hat{\theta}_T = \hat{\theta}_T^k \)
Parameter choice: Interval set

for every $\tau$ with the following interval lengths:

$$\{s_k\}_{k=1}^{13} = \{1w, 1m, 3m, 6m, 1y, 1.5y, 2y, 2.5y, 3y, 3.5y, 4y, 4.5y, 5y\},$$

where $w$ denotes a week (5 days), $m$ refers to one month (21 days) and $y$ to one year (252 days).
Parameter choice: CV

Monte Carlo simulation: generate AR(p) processes with
\( \theta_t = \theta^* = (\theta_0, \theta_1, \ldots, \theta_p, \sigma) \) for all \( t \). For example AR(1) processes:

\[
\begin{align*}
\text{y}_t &= \theta^*_1 + \theta^*_2 \text{y}_{t-h} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \theta^*_3^2) \\
\text{y}_0 &= \theta^*_1 / (1 - \theta^*_2) \\
\end{align*}
\]

100 000 paths, each including 1261 observations

Choice of critical values:

Parametric case \( \theta_t \equiv \theta^* \):

\[
E_{\theta^*} \left| LR \left( I^K, \tilde{\theta}_t^K, \hat{\theta}_t^K(\zeta_1, \ldots, \zeta_K) \right) \right|^r \leq \xi_r \quad (3)
\]

\[
E_{\theta^*} \left| LR \left( I^k, \tilde{\theta}_t^k, \hat{\theta}_t^k(\zeta_1, \ldots, \zeta_k) \right) \right|^r \leq \frac{k - 1}{K - 1} \xi_r \quad (4)
\]
Localized realized volatility

Parameter choice: CV

Sequential choice of critical values

☐ Choice of $\zeta_1 = \infty$ initializing the procedure $\hat{\theta}_t^1(\zeta_1) = \tilde{\theta}_t^1$

☐ Choice of $\zeta_2$ leading to $\hat{\theta}_t^k(\zeta_1, \zeta_2)$ by setting $\zeta_3 = \ldots = \zeta_K = \infty$:

$$E_{\theta^*}|LR(I^k, \tilde{\theta}_t^k, \hat{\theta}_t^k(\zeta_1, \zeta_2))|^r \leq \frac{1}{K-1}\xi_r, \quad k = 2, \ldots, K.$$
Localized realized volatility

Parameter choice: CV

Sequential choice of critical values

- Choice of $\zeta_3$ leading to $\hat{\theta}^k_t(\zeta_1, \zeta_2, \zeta_3)$ by setting $\zeta_4 = \ldots = \zeta_K = \infty$:

$$E_{\theta^*} \left| LR \left( I^k, \tilde{\theta}^k_t, \hat{\theta}^k_t(\zeta_1, \zeta_2, \zeta_3) \right) \right|^r \leq \frac{2}{K - 1} \xi_r, \quad k = 3, \ldots, K.$$ 

- Choice of $\zeta_k$ leading to $\hat{\theta}^l_t(\zeta_1, \ldots, \zeta_k)$ by setting $\zeta_{k+1} = \ldots = \zeta_K = \infty$:

$$E_{\theta^*} \left| LR \left( I^k, \tilde{\theta}^l_t, \hat{\theta}^l_t(\zeta_1, \ldots, \zeta_k) \right) \right|^r \leq \frac{k - 1}{K - 1} \xi_r, \quad l = k, \ldots, K.$$
Parameter choice: CV and $r$

- The critical values (CV) depend on $\theta^*$ used in the Monte Carlo simulation:
  - local global CV: global constant parameter $\theta^*$ over the whole sample
  - local adaptive CV: time varying parameter $\theta^*$ using a moving window with fixed size.

- $r$: default choice 1/2
Critical values

Figure 4: CV for LAR(1) model with $r = 1/2$ and $\theta^* = (-0.116, 0.783, 0.553)^\top$ (global). Data source: Log RV of the S&P500 index futures.
Simulation on LAR(1)

Figure 5: Simulation results for scenarios S1 (changing parameter: $\theta_{0t}$).
Simulation on LAR(1)

Figure 6: Simulation results for scenarios G1 (changing parameter: $\theta_{0t}$).
Simulation on LAR(1)

Figure 7: Simulation results for scenarios S2 (changing parameter: $\theta_{1t}$).
Simulation on LAR(1)

Figure 8: Simulation results for scenarios G2 (changing parameter: $\theta_{1t}$).
Simulation on LAR(1)

Figure 9: Simulation results for scenarios S3 (changing parameter: $\sigma_t$).
Simulation on LAR(1)

Figure 10: Simulation results for scenarios G3 (changing parameter: $\sigma_t$).
## Sensitivity analysis: Impact of parameters

<table>
<thead>
<tr>
<th>Choice of parameters</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K = 9$</td>
<td>$K = 22$</td>
<td>$r = 1/3$</td>
<td>$r = 1$</td>
</tr>
<tr>
<td>$0.8 \theta^*$</td>
<td>0.9974</td>
<td>0.9726</td>
<td>1.0395</td>
<td></td>
</tr>
<tr>
<td>$1.2 \theta^*$</td>
<td>0.9974</td>
<td>0.9726</td>
<td>1.0395</td>
<td></td>
</tr>
</tbody>
</table>

### Scenario S2

<table>
<thead>
<tr>
<th>R-RMSFE</th>
<th>0.9956</th>
<th>1.0128</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-DS</td>
<td>50% 75%</td>
<td>50% 75%</td>
</tr>
<tr>
<td>$t = 1501$</td>
<td>-1 -1</td>
<td>2 2</td>
</tr>
<tr>
<td>$t = 2001$</td>
<td>0 0</td>
<td>3 0</td>
</tr>
<tr>
<td>$t = 2401$</td>
<td>0 -</td>
<td>5 -</td>
</tr>
<tr>
<td>$t = 2801$</td>
<td>&gt;344 -</td>
<td>&gt;344 -</td>
</tr>
</tbody>
</table>

### Scenario G2

<table>
<thead>
<tr>
<th>R-RMSFE</th>
<th>0.9946</th>
<th>1.0132</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-DS</td>
<td>50% 75%</td>
<td>50% 75%</td>
</tr>
<tr>
<td>$t = 1601$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$t = 2001$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$t = 2501$</td>
<td>0 -</td>
<td>0 -</td>
</tr>
<tr>
<td>$t = 2801$</td>
<td>&gt;399 &gt;54</td>
<td>&gt;399 &gt;54</td>
</tr>
</tbody>
</table>

Reported are the relative one-step-ahead RMSFEs and the relative detections speeds (R-DS) in the scenarios S2 and G2 for different choices of the parameters.
## Sensitivity analysis: Model misspecification

<table>
<thead>
<tr>
<th>DGP:</th>
<th>local const.</th>
<th>LAR(2)</th>
<th>LAR(5)</th>
<th>LAR(10)</th>
<th>ARFIMA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\text{DGP}} )</td>
<td>( \theta_{0t} )</td>
<td>1.0225</td>
<td>0.6247</td>
<td>0.5664</td>
<td>0.5568</td>
</tr>
<tr>
<td>LAR(1)</td>
<td>( \theta_{2t} )</td>
<td>0.9339</td>
<td>0.6293</td>
<td>0.5848</td>
<td>0.5724</td>
</tr>
</tbody>
</table>

Table 2: Reported are the average RMSFEs based on the LAR(1) procedure and the estimated data generating processes, \( \hat{\text{DGP}} \).
The ARFIMA model

The autoregressive fractionally integrated moving average, ARFIMA\((p, d, q)\), model:

\[
\phi(L)(1 - L)^d(\log RV_t - \mu) = \psi(L)u_t,
\]

with \(\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p, \psi(L) = 1 + \psi_1 L + \cdots + \psi_q L^q\), \(L\) denoting the lag operator, \(d \in (0, 0.5)\) and \(u_t \overset{iid}{\sim} N(0, \sigma^2)\), see e.g. Andersen, Bollerslev, Diebold and Labys (2003).
The HAR model

The heterogeneous autoregressive, HAR, model:

\[
\log RV_t = \alpha_0 + \alpha_d \log RV_{t-1} + \alpha_w \log RV_{t-5:t-1} \\
+ \alpha_m \log RV_{t-21:t-1} + u_t
\]

with the multiperiod realized volatility components defined by

\[
RV_{t+1-k:t} = \frac{1}{k} \sum_{j=1}^{k} RV_{t-j}
\]

and \( u_t \overset{iid}{\sim} N(0, \sigma^2) \), see Corsi (2004).
Empirical evidence

- Although the HAR model is formally no long memory model, it seems to provide a good approximation of the long range dependence.
- The HAR and ARFIMA models exhibit similar in-sample and out-of-sample performance.
- Both strongly outperform conventional volatility models.
Forecast setup

- The first five years of the S&P 500 index futures data serve as *training set*.
- The remaining data serves as *forecast evaluation period* (January 2, 1990 to February 4, 2005).
Forecast setup

 Setup for LAR(1) models:

- Consider 5 sets of critical values: the global ones and the adaptive ones based on a sample period of 1 month, 6 months, 1 year and 2.5 years.

- The interval of homogeneity is always selected based on the following set of interval lengths:

\[
\{s_k\}^{13}_{k=1} = \{1w, 1m, 3m, 6m, 1y, 1.5y, 2y, 2.5y, 3y, 3.5y, 4y, 4.5y, 5y\}
\]}
Figure 11: Boxplot of the homogenous intervals selected by the LAR(1) procedure based on different sets of critical values.
Forecast setup

Setup for ARFIMA, HAR and AR(1) models:

- All forecasts are based on an ARFIMA(2, $d$, 0) model (as indicated by the AIC and BIC using the full sample period).
- Estimation and prediction is performed using a rolling window scheme with different window sizes, i.e. 
  \{3m, 6m, 1y, 1.5y, 2y, 2.5y, 3y, 3.5y, 4y, 4.5y, 5y\}.
- Same setup is used to assess the predictability of the HAR model and a constant AR(1) model, i.e.
  \[
  \log RV_t = \alpha_0 + \alpha_1 \log RV_{t-1} + u_t \quad \text{with} \quad u_t \overset{iid}{\sim} N(0, \sigma^2).
  \]
Figure 12: one-day-ahead prediction.
Empirical analysis

![Financial events & selected intervals](image)

**Figure 13: Financial events & selected intervals.**
<table>
<thead>
<tr>
<th>crit. values</th>
<th>LAR(1)</th>
<th>sample size</th>
<th>AR(1)</th>
<th>ARFIMA</th>
<th>HAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>local adaptive 1m</td>
<td>0.4858</td>
<td>3m</td>
<td><strong>0.5149</strong></td>
<td>0.5328</td>
<td>0.5381</td>
</tr>
<tr>
<td>local adaptive 6m</td>
<td><strong>0.4811</strong></td>
<td>6m</td>
<td>0.5288</td>
<td>0.5225</td>
<td>0.5240</td>
</tr>
<tr>
<td>local adaptive 1y</td>
<td>0.4876</td>
<td>1y</td>
<td>0.5398</td>
<td>0.5178</td>
<td>0.5185</td>
</tr>
<tr>
<td>local adaptive 2.5y</td>
<td>0.4916</td>
<td>1.5y</td>
<td>0.5462</td>
<td>0.5143</td>
<td>0.5172</td>
</tr>
<tr>
<td>local global</td>
<td>0.5014</td>
<td>2y</td>
<td>0.5509</td>
<td>0.5133</td>
<td>0.5158</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.5y</td>
<td>0.5555</td>
<td>0.5132</td>
<td><strong>0.5153</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3y</td>
<td>0.5574</td>
<td><strong>0.5123</strong></td>
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<td>3.5y</td>
<td>0.5607</td>
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<td></td>
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<td>4y</td>
<td>0.5649</td>
<td>0.5129</td>
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<td></td>
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<td>4.5y</td>
<td>0.5686</td>
<td>0.5130</td>
<td>0.5173</td>
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<td></td>
<td></td>
<td>5y</td>
<td>0.5712</td>
<td>0.5129</td>
<td>0.5176</td>
</tr>
</tbody>
</table>

Table 3: Root mean square forecast errors.
<table>
<thead>
<tr>
<th>crit. values</th>
<th>LAR(1)</th>
<th>sample size</th>
<th>AR(1)</th>
<th>ARFIMA</th>
<th>HAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>local adaptive 1m</td>
<td>0.3667</td>
<td>3m</td>
<td>0.3900</td>
<td>0.3978</td>
<td>0.4025</td>
</tr>
<tr>
<td>local adaptive 6m</td>
<td>0.3654</td>
<td>6m</td>
<td>0.3987</td>
<td>0.3902</td>
<td>0.3862</td>
</tr>
<tr>
<td>local adaptive 1y</td>
<td>0.3704</td>
<td>1y</td>
<td>0.4057</td>
<td>0.3860</td>
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<tr>
<td>local adaptive 2.5y</td>
<td>0.3748</td>
<td>1.5y</td>
<td>0.4103</td>
<td>0.3836</td>
<td>0.3843</td>
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<td>local global</td>
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<td>2y</td>
<td>0.4136</td>
<td>0.3826</td>
<td>0.3836</td>
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<td></td>
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<td>2.5y</td>
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<td>4y</td>
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<td>5y</td>
<td>0.4300</td>
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<td>0.3858</td>
</tr>
</tbody>
</table>

Table 4: Mean absolute forecast error.
Empirical results

- **Mincer–Zarnowitz regression:**
  Evaluate the forecasting performance of the different models based on Mincer–Zarnowitz regressions:

  \[
  \log RV_t = \alpha + \beta \widehat{\log RV}_{t,i} + \nu_t
  \]

  with \( \widehat{\log RV}_{t,i} \) denoting the log. realized volatility forecast of model \( i \).

  - Assess \( R^2 \) of the regression.
  - Test on *unbiasedness* of the different forecasts:
    \( H_0 : \alpha = 0 \) and \( \beta = 1 \).
<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>p-value</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>global LAR</td>
<td>$-0.0130$</td>
<td>$1.0128$</td>
<td>$0.1007$</td>
<td>$0.6959$</td>
</tr>
<tr>
<td></td>
<td>$(0.0125)$</td>
<td>$(0.0142)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>adaptive LAR, 1y</td>
<td>$0.0025$</td>
<td>$1.0014$</td>
<td>$0.9780$</td>
<td>$0.7117$</td>
</tr>
<tr>
<td></td>
<td>$(0.0123)$</td>
<td>$(0.0127)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1y AR(1)</td>
<td>$-0.0010$</td>
<td>$1.0117$</td>
<td>$0.6002$</td>
<td>$0.4669$</td>
</tr>
<tr>
<td></td>
<td>$(0.0144)$</td>
<td>$(0.0158)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5y AR(1)</td>
<td>$0.0221$</td>
<td>$1.0367$</td>
<td>$0.2216$</td>
<td>$0.6052$</td>
</tr>
<tr>
<td></td>
<td>$(0.0162)$</td>
<td>$(0.0213)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1y ARFIMA</td>
<td>$0.0008$</td>
<td>$1.0011$</td>
<td>$0.9962$</td>
<td>$0.6747$</td>
</tr>
<tr>
<td></td>
<td>$(0.0119)$</td>
<td>$(0.0132)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5y ARFIMA</td>
<td>$0.0009$</td>
<td>$1.0154$</td>
<td>$0.4907$</td>
<td>$0.6811$</td>
</tr>
<tr>
<td></td>
<td>$(0.0115)$</td>
<td>$(0.0129)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1y HAR</td>
<td>$-0.0076$</td>
<td>$0.9907$</td>
<td>$0.7509$</td>
<td>$0.6742$</td>
</tr>
<tr>
<td></td>
<td>$(0.0128)$</td>
<td>$(0.0128)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5y HAR</td>
<td>$0.0145$</td>
<td>$1.0237$</td>
<td>$0.2036$</td>
<td>$0.6756$</td>
</tr>
<tr>
<td></td>
<td>$(0.0119)$</td>
<td>$(0.0133)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Mincer–Zarnowitz regression results.
Empirical results

Test on equal forecast performance:
Diebold–Mariano test on equal MSFEs:

\[ e_{t,LAR}^2 - e_{t,i}^2 = \mu + \nu_t \]

with \( e_{t,i} \) denoting the forecast error of model \( i \).

\[ H_0 : \mu = 0. \]
### Table 6: Diebold–Mariano test results.

<table>
<thead>
<tr>
<th>Global LAR compared to</th>
<th>$\mu$</th>
<th>Adaptive LAR, 1y compared to</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1y AR(1)</td>
<td>$-0.0400$</td>
<td>1y AR(1)</td>
<td>$-0.0535$</td>
</tr>
<tr>
<td></td>
<td>$(0.0096)$</td>
<td></td>
<td>$(0.0097)$</td>
</tr>
<tr>
<td>5y AR(1)</td>
<td>$-0.0141$</td>
<td>5y AR(1)</td>
<td>$-0.0546$</td>
</tr>
<tr>
<td></td>
<td>$(0.0109)$</td>
<td></td>
<td>$(0.0111)$</td>
</tr>
<tr>
<td>1y ARFIMA</td>
<td>$-0.0168$</td>
<td>1y ARFIMA</td>
<td>$-0.0304$</td>
</tr>
<tr>
<td></td>
<td>$(0.0109)$</td>
<td></td>
<td>$(0.0108)$</td>
</tr>
<tr>
<td>5y ARFIMA</td>
<td>$-0.0118$</td>
<td>5y ARFIMA</td>
<td>$-0.0253$</td>
</tr>
<tr>
<td></td>
<td>$(0.0109)$</td>
<td></td>
<td>$(0.0109)$</td>
</tr>
<tr>
<td>1y HAR</td>
<td>$-0.0175$</td>
<td>1y HAR</td>
<td>$-0.0310$</td>
</tr>
<tr>
<td></td>
<td>$(0.0104)$</td>
<td></td>
<td>$(0.0103)$</td>
</tr>
<tr>
<td>5y HAR</td>
<td>$-0.0165$</td>
<td>5y HAR</td>
<td>$-0.0258$</td>
</tr>
<tr>
<td></td>
<td>$(0.0104)$</td>
<td></td>
<td>$(0.0103)$</td>
</tr>
</tbody>
</table>
Long-term forecast setup

- Setup for LAR(1) and AR*(1) model

\[
\begin{align*}
\log RV_{t+10} &= \theta_1 + \theta_2 \log RV_t + \varepsilon_t^* \\
\log RV_{t+10} &= \theta_1 t + \theta_2 t \log RV_t + \varepsilon_t
\end{align*}
\]

- Consider 5 sets of critical values: the global ones and the adaptive ones based on a sample period of 1 month, 6 months, 1 year and 2.5 years.

- The interval of homogeneity is always selected based on the following set of interval lengths:

\[
\{s_k\}_{k=1}^{13} = \{1w, 1m, 3m, 6m, 1y, 1.5y, 2y, 2.5y, 3y, 3.5y, 4y, 4.5y, 5y\}
\]
Figure 14: 10-day-ahead prediction.
## Table 7: Root mean square forecast errors for 10-day-ahead prediction.

<table>
<thead>
<tr>
<th>Crit. values</th>
<th>LAR(10)</th>
<th>Sample size</th>
<th>AR*(1)</th>
<th>ARFIMA</th>
<th>HAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local adaptive 1m</td>
<td>0.4615</td>
<td>3m</td>
<td>0.5873</td>
<td>0.6463</td>
<td>0.7375</td>
</tr>
<tr>
<td>Local adaptive 6m</td>
<td>0.4873</td>
<td>6m</td>
<td>0.6115</td>
<td>0.6352</td>
<td>0.6581</td>
</tr>
<tr>
<td>Local adaptive 1y</td>
<td>0.4945</td>
<td>1y</td>
<td>0.6282</td>
<td>0.6286</td>
<td>0.6349</td>
</tr>
<tr>
<td>Local adaptive 2.5y</td>
<td>0.5056</td>
<td>1.5y</td>
<td>0.6399</td>
<td>0.6214</td>
<td>0.6263</td>
</tr>
<tr>
<td>Local global</td>
<td>0.5884</td>
<td>2y</td>
<td>0.6504</td>
<td>0.6235</td>
<td>0.6249</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.5y</td>
<td>0.6559</td>
<td>0.6214</td>
<td>0.6232</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3y</td>
<td>0.6623</td>
<td>0.6207</td>
<td>0.6237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.5y</td>
<td>0.6678</td>
<td>0.6215</td>
<td>0.6252</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4y</td>
<td>0.6751</td>
<td>0.6214</td>
<td>0.6270</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.5y</td>
<td>0.6817</td>
<td>0.6222</td>
<td>0.6281</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5y</td>
<td>0.6888</td>
<td>0.6240</td>
<td>0.6305</td>
</tr>
</tbody>
</table>
Conclusion

- Investigate the dual view on the long memory diagnosis of volatility.
- The long memory phenomenon can alternatively be described by parsimonious short memory models with structural breaks.
- Identification of structural breaks by a localized realized volatility model.
Conclusion

The empirical results show that:

- the localized approach outperforms long memory-type models and constant AR models in terms of predictability.
- an adaptive choice of the critical values (and a decrease in the underlying sample period) improves the estimation and forecast accuracy of the localized approach.
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