2014-04-16 10:28

Working Paper

An Analytic Derivation and Comments on Betting against Beta

Deng-Ta Chen

Deng-Ta CHEN is from WiSE (Wang Yanan Institute for Studies in Economics), Xiamen University.

Deng-Ta CHEN's Email: MAXCHEN@XMU.EDU.CN, Fax: 86+592-218 7708.

This version: 2014-04-16

2013-12-28, add Figure 1

2013-12-10, first version

Electronic copy available at: http://ssrn.com/abstract=2426056

An Analytic Derivation and Comments on Betting against Beta*

Abstract: I present a closed form solution to the OLG model in Frazzini and Pedersen (2014), and find that their OLG model of heterogeneous investors results in the zero-beta CAPM. I prove that the optimal amount to invest in risky assets for an investor is determined exactly by her risk aversion, margin requirement and wealth. Thus every investor's optimal portfolio is explicitly solved and the market portfolio is easily aggregated on the mean-variance frontier. Working with the analytic formulas, I provide a rigorous derivation of their theoretical results and discuss the implications of the Z-factor, which is market neutral, and must be the second factor other than the market factor in capital asset pricing.

Keywords: Zero-beta Portfolio, CAPM, Heterogeneous Investors, Z-factor

JEL Classification: G12; G11

---

*I thank Cheng Hsiao for helpful suggestions. I am grateful for valuable discussions with Qiang Chen and Terry Tsai.

Electronic copy available at: http://ssrn.com/abstract=2426056

In a recent open access paper of JFE (Journal of Financial Economics, Volume 111, Issue 1, January 2014), Frazzini and Pedersen (2014, henceforth FP) employ an OLG (overlapping-generations) economy to model asset price dynamics with heterogeneous investors subject to leverage and margin constraints. They conduct a remarkable empirical study on the model's five central implications for US equities, 20 international equity markets, Treasury bonds, corporate bonds, and futures. However, without reaching a final closed-form solution, FP make use of the Lagrange multiplier and conclude that (from FP's abstract)

1. Because constrained investors bid up high-beta assets, high beta is associated with low alpha.
2. A betting against beta (BAB) factor[1], which is long leveraged low-beta assets and short high-beta assets, produces significant positive risk-adjusted returns.
3. When funding constraints tighten, the return of the BAB factor is low.
4. Increased funding liquidity risk compresses betas toward one.
5. More constrained investors hold riskier assets.

To validate FP's propositions, I solve their OLG model analytically. I show that the first conclusion is the fundamental result, and provide a more clear statement and a rigorous proof. However, with respect to other predictions, different implications could follow. Thus, I carry out a close examination and obtain more details theoretically.

The remainder of this article is organized as follows. Section 1 is an illustration of the main logic of FP's theory. Section 2 presents a closed-form solution to the OLG model of FP in mean-variance space. Section 3 revisits FP's propositions. Section 4 concludes the paper. In addition, appendix A reviews relevant properties of the mean-variance frontier, which is essential for understanding the solutions to FP's OLG model.

1 An Illustration

FP's OLG model is built in the payoff space, nevertheless, it is much easier to understand in the mean-variance space. The fundamental result of FP is intuitively depicted in Figure 1. Due

[1] In fact, the BAB factor is the beta factor in Black (1993, p10): "We can construct the beta factor by creating a diversified portfolio that is long in low-beta stocks and short in smaller amounts of high-beta stocks, so that its beta is roughly zero."

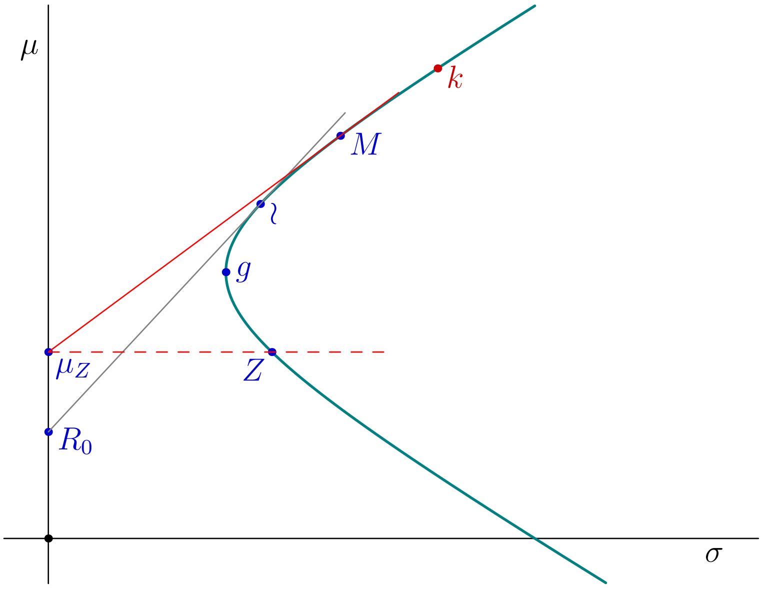

Figure 1: Portfolio Choice with Constraints

In the mean-standard deviation ($\mu - \sigma$) plane: unconstrained investors prefer to invest in the tangent portfolio $\wr$, to achieve the highest Sharpe ratio. However, constrained investors seek higher expected return than the tangent portfolio, for example, portfolio k. As a result, the market portfolio M has a higher expected return than the tangent portfolio $\wr$, and the zero-beta portfolio of market portfolio has the expected return μ_Z greater than the risk-free rate R_0.

to leverage and margin constraints, there are two types of investors, i.e., constrained investors and unconstrained investors. For unconstrained investors, the optimal portfolio is the tangent portfolio $\wr$, which has the highest expected excess return (with respect to the riskless rate R_0) per unit of risk. For constrained investors, the funding constraint is binding, thus they keep the minimum amount of money in cash, and put the rest in the risky assets. In the trade-off between risk and return to achieve maximum utility, they choose portfolios of higher expected returns above the tangent portfolio. In Figure 1, one of the constrained investors holds portfolio k.

The market portfolio M is the weighted average of all investors' portfolios, thus, in the equilibrium of FP's OLG model, the market portfolio is no longer the tangent portfolio as in the standard CAPM. The market portfolio is riskier than the tangent portfolio with a lower Sharpe ratio. Because the disparity of the market portfolio and the tangent portfolio, the zero-beta CAPM be viewed as

$$\mu_A = R_0 + \beta_A(\mu_M - R_0) + (1 - \beta_A)(\mu_Z - R_0),$$

where μ_A is the expected return of any risky asset or portfolio, β_A is the traditional beta, μ_M is the expected return of market portfolio, and μ_Z is the expected return of the zero-beta portfolio of the market portfolio. Since the market portfolio has a higher expected return than the tangent portfolio, μ_Z is greater than the riskless rate R_0.

In this market of heterogeneous investors, there are two factors playing a role in capital asset pricing, i.e., the market factor R_M and the zero beta portfolio of market portfolio—the Z-factor R_Z. To capture the Z-factor, FP construct a BAB (betting against beta) factor, which has zero beta like Z-factor, but has a different expected return. In general, the BAB factor is not efficient, and is not lying on the horizontal dotted line passing through Z in Figure 1.

2 Heterogeneous Investors

FP define their overlapping-generations (OLG) model as

$$\begin{aligned} \max \quad & x^{i\prime}\left(\mathrm{E}_t(P_{t+1}+\delta_{t+1})-(1+r^f)P_t\right)-\frac{\gamma^i}{2}x^{i\prime}\Omega_t x^i \\ \text{s.t.} \quad & x^{i\prime}P_t \leqslant W_t^i/m_t^i \end{aligned} \tag{2.1}$$

where notations are defined in their paper.

To avoid confusions on superscript (which usually denotes exponent), I rewrite agent i's risk aversion $\gamma_i = \gamma_t^i > 0$, margin requirement $m_i = m_t^i$, and wealth $W_i = W_t^i$ (subscript t is dropped when it is clear from the context). Furthermore, for the sake of clarity, I use bold font for vectors and matrixes, and thus denote portfolio of shares $\mathbf{h}_i = x^i$, payoff $\mathbf{x}_{t+1} = P_{t+1}+\delta_{t+1}$, variance matrix $\boldsymbol{\Omega} = \Omega_t$, and current price $\mathbf{p}_t = P_t$. Let $\mathbf{P} = \mathrm{diag}(\mathbf{p}_t)$, then the vector of gross returns is $\mathbf{r}_{t+1} = \mathbf{P}^{-1}\mathbf{x}_{t+1}$, the mean vector is

$$\boldsymbol{\mu} = \mathrm{E}_t(\mathbf{r}) = \mathbf{P}^{-1}\,\mathrm{E}_t(P_{t+1}+\delta_{t+1}),$$

and the variance matrix is

$$\mathbf{V} = \mathrm{var}_t(\mathbf{r}) = \mathbf{P}^{-1}\boldsymbol{\Omega}\mathbf{P}^{-1}.$$

Applying the new notations, model (2.1) becomes

$$\begin{aligned} \max \quad & U_i = \mathbf{h}_i'\mathbf{P}\boldsymbol{\mu} - R_0\mathbf{h}_i'\mathbf{p}_t - \frac{\gamma_i}{2}\mathbf{h}_i'\mathbf{PVPh}_i \\ \text{s.t.} \quad & \mathbf{h}_i'\mathbf{p}_t \leqslant W_i/m_i \end{aligned} \tag{2.2}$$

2.1 The Funding Constraint

The optimal solution to model (2.2) may be on the boundary or interior. Given risk aversion γ_i, margin requirement m_i and wealth W_i, is it possible to identify whether the optimal solution is on the boundary or not? To address this issue, fix the amount of wealth w, and consider the following model[2]

$$\begin{aligned} \max \quad & U = \mathbf{h}_i' \mathbf{P} \boldsymbol{\mu} - R_0 w - \frac{\gamma_i}{2} \mathbf{h}_i' \mathbf{PVPh}_i \\ \text{s.t.} \quad & \mathbf{h}_i' \mathbf{p}_t = w \end{aligned} \tag{2.3}$$

Define the Lagrange function

$$L = \mathbf{h}_i' \mathbf{P} \boldsymbol{\mu} - R_0 w - \frac{\gamma_i}{2} \mathbf{h}_i' \mathbf{PVPh}_i - \psi_i \left(\mathbf{h}_i' \mathbf{p}_t - w \right).$$

Using the notations defined in appendix A, it is a simple exercise to find the optimal solution

$$\mathbf{h}_i = \frac{1}{\gamma_i} \mathbf{P}^{-1} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - \frac{b - \gamma_i w}{a} \mathbf{1} \right),$$

with Lagrange multiplier[3]

$$\psi_i = \frac{b - \gamma_i w}{a} - R_0. \tag{2.4}$$

The maximum utility of model (2.3) is a quadratic function of w

$$U = -\frac{1}{2a\gamma_i} \left(\gamma_i^2 w^2 + 2\gamma_i \left(a R_0 - b \right) w + \left(b^2 - ac \right) \right).$$

Setting $\frac{\mathrm{d}U}{\mathrm{d}w} = 0$, we get

$$w = \frac{1}{\gamma_i} \left(b - a R_0 \right),$$

which shows that if $w > \frac{1}{\gamma_i} \left(b - a R_0 \right)$, the utility is decreasing with w, and if $w < \frac{1}{\gamma_i} \left(b - a R_0 \right)$, the utility is increasing with w. Thus, based on the analysis of model (2.3), we have the following result for model (2.2)

Lemma 1. *On the optimal solution to model (2.2)*

(i) If $\frac{W_i}{m_i} \leqslant \frac{1}{\gamma_i} \left(b - a R_0 \right)$, then the funding constraint is binding, $\mathbf{h}_i' \mathbf{p}_t = \frac{W_i}{m_i}$.

(ii) If $\frac{W_i}{m_i} > \frac{1}{\gamma_i} \left(b - a R_0 \right)$, then the funding constraint is not binding, $\mathbf{h}_i' \mathbf{p}_t = \frac{1}{\gamma_i} \left(b - a R_0 \right)$.

[2] It can be shown that, model (2.3) amounts to the *basic problem* of Sharpe (1970, p59).

[3] In fact, here $\psi_i = \frac{b - \gamma_i w}{a}$. However, to match FP's Lagrange multiplier, writing equivalently

$$L = \mathbf{h}_i' \mathbf{P} \boldsymbol{\mu} - R_0 \mathbf{h}_i' \mathbf{p}_t - \frac{\gamma_i}{2} \mathbf{h}_i' \mathbf{PVPh}_i - \psi_i \left(\mathbf{h}_i' \mathbf{p}_t - w \right)$$

which produces equation (2.4).

From Lemma 1, the optimal amount to invest in risky assets for investor i is

$$w_i = \min\left(\frac{W_i}{m_i}, \frac{1}{\gamma_i}(b - aR_0)\right) > 0, \tag{2.5}$$

and the vector of optimal portfolio weights is

$$\mathbf{z}_i = \frac{1}{w_i}\mathbf{P}\mathbf{h}_i = \frac{1}{u_i}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - \frac{b}{a}\mathbf{1}\right) + \frac{1}{a}\mathbf{V}^{-1}\mathbf{1},$$

where

$$u_i = \gamma_i w_i > 0.$$

Thus

$$\begin{aligned} \mu_i &= \mathbf{z}_i'\boldsymbol{\mu} = \frac{ac - b^2}{au_i} + \frac{b}{a} > \mu_g, \\ v_i &= \mathbf{z}_i'\mathbf{V}\mathbf{z}_i = \frac{ac - b^2}{au_i^2} + \frac{1}{a} = \frac{h(\mu_i)}{ac - b^2}. \end{aligned} \tag{2.6}$$

From equation (A.1), obviously $\mathbf{z}_i \in \mathsf{F}$, and investor i invests in an efficient frontier portfolio since its expected return is higher than that of the global minimum variance portfolio.

For the constrained investors and unconstrained investors, currently we are at right time to examine the portfolio positions in Figure 1: unconstrained investors invest $w_i = \frac{1}{\gamma_i}(b - aR_0)$ in risky assets

$$u_i = \gamma_i w_i = b - aR_0,$$

thus by equation (2.6)

$$\mu_i = \frac{c - bR_0}{b - aR_0} = \mu_\wr.$$

Unconstrained investors prefer to invest in the tangent portfolio to achieve the highest expected excess return per unit of risk (Sharpe ratio). However, for constrained investors, $w_k = \frac{W_k}{m_k} \leqslant \frac{1}{\gamma_k}(b - aR_0)$, thus

$$u_k = \gamma_k w_k \leqslant b - aR_0.$$

Following equation (2.6), constrained investors invest their limited money in riskier portfolios other than the tangent portfolio

$$\mu_k \geqslant \mu_\wr,$$

to achieve a higher expected return with a higher risk. In the following discussion, I assume that at least one constrained investor seeks higher expected return than that of the tangent portfolio, i.e., $\mu_k > \mu_\wr$ strictly.

2.2 The Market Portfolio

The total market value of the risky assets, $w_M = \sum_i w_i$, is the summation over investors' optimal value of risky assets. The market portfolio R_M has the weights

$$\mathbf{z}_M = \frac{1}{w_M}\sum_i w_i \mathbf{z}_i = \frac{1}{u_M}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - \frac{b}{a}\mathbf{1}\right) + \frac{1}{a}\mathbf{V}^{-1}\mathbf{1},$$

where

$$u_M = w_M \gamma,$$

and $\frac{1}{\gamma} = \sum_i \frac{1}{\gamma_i}$. The market portfolio R_M is a convex combination of frontier portfolios, it is a frontier portfolio

$$\begin{aligned} \mu_M &= \mathbf{z}_M' \boldsymbol{\mu} = \frac{ac - b^2}{a u_M} + \frac{b}{a}, \\ v_M &= \mathbf{z}_M' \mathbf{V} \mathbf{z}_M = \frac{ac - b^2}{a u_M^2} + \frac{1}{a} = \frac{h(\mu_M)}{ac - b^2}. \end{aligned} \tag{2.7}$$

The market portfolio R_M has a higher expected return than the tangent portfolio R_l, and thus it is riskier. Since the market portfolio is the weighted average of all investors' portfolios, where unconstrained investors invest in the tangent portfolio and constrained investors prefer riskier portfolios above the tangent portfolio for higher expected returns. Hence, as the graph in Figure 1, the zero-beta portfolio of market portfolio has the expected return greater than the risk-free rate

$$\mu_Z = \frac{b - u_M}{a} > R_0. \tag{2.8}$$

2.3 The Weighted Average Lagrange Multiplier

The weighted average Lagrange multiplier in FP is defined as

$$\psi \equiv \sum \frac{\gamma}{\gamma_i}\psi_i,$$

where ψ_i is defined in equation (2.4) with $w = w_i$, thus

$$\psi = \frac{b - u_M}{a} - R_0 = \mu_Z - R_0 > 0. \tag{2.9}$$

From equation (2.5)

$$u_M = w_M \gamma = \frac{\sum_i w_i}{\sum_i \frac{1}{\gamma_i}} = \frac{\sum_i \min\left(\frac{W_i}{m_i}, \frac{1}{\gamma_i}(b - aR_0)\right)}{\sum_i \frac{1}{\gamma_i}},$$

when the funding constraint is binding, $w_i = \frac{W_i}{m_i} \leqslant \frac{1}{\gamma_i}(b - aR_0)$, the tighter the funding constraint by increasing the m_i, will induce a lower u_M, and thus a higher ψ. This is why FP regard the average Lagrange multiplier ψ as the funding tightness. It is worthy to mention that ψ is affected not only from the attributes (wealth W_i, margin requirement m_i and risk aversion γ_i) of agents, but also from the settings (mean vector $\boldsymbol{\mu}$, variance matrix $\mathbf{V}$, and risk-free rate R_0) of the market.

3 Revisiting FP's Propositions

FP's five central predictions are derived from their propositions. However, their proofs are inadequate due to the lack of a closed form solution. In this section, I make use of the connection between the mean-variance space and the payoff space, and obtain mathematical evidence to revisit their propositions.

3.1 Proposition 1

Proposition 1. *(high beta is low alpha).*

(i) The equilibrium required return for any security s follows equation (A.3), the zero-beta CAPM

$$\mu_s = \mu_Z + \beta_s(\mu_M - \mu_Z). \tag{3.1}$$

(ii) A security's alpha with respect to the market is $\alpha_s = (1 - \beta_s)\mu_Z$. The alpha decreases in the beta.

(iii) For an efficient portfolio, the Sharpe ratio is highest for an efficient portfolio with a beta less than one.

The beta vector is computed as

$$\boldsymbol{\beta} = \frac{\text{cov}\left(\mathbf{r}, R_M\right)}{\text{var}\left(R_M\right)} = \frac{\mathbf{V}\mathbf{z}_M}{v_M} = \frac{u_M}{ac - b^2 + u_M^2}\left(a\boldsymbol{\mu} + \left(u_M - b\right)\mathbf{1}\right), \tag{3.2}$$

simplifying by equation (2.7) and (2.8)

$$\boldsymbol{\beta} = \frac{1}{\mu_M - \mu_Z}\left(\boldsymbol{\mu} - \mu_Z\mathbf{1}\right),$$

thus

$$\boldsymbol{\mu} = \mu_Z\mathbf{1} + (\mu_M - \mu_Z)\boldsymbol{\beta},$$

which states that security s follows equation (3.1). Because of $\mu_Z = \psi + R_0$ in equation (2.9)

$$\mu_s = \psi + R_0 + \beta_s(\mu_M - \psi - R_0),$$

which replicates the equation (8) of FP.

In light of equation (3.1), a security's alpha with respect to the market is $\alpha_s = (1 - \beta_s)\mu_Z$. Surely, the alpha decreases in the beta for $\mu_Z > R_0 > 0$. Theoretically, R_Z is another risk factor, which has zero beta with respect to the market portfolio R_M, and thus is market neutral. It is unfortunate that *the Z-factor*, R_Z, *has been shielded by alpha for half a century*.

The efficient portfolio with the highest Sharpe ratio is the tangent portfolio. Since

$$\frac{\mathrm{d}\beta}{\mathrm{d}\mu} = \frac{1}{\mu_M - \mu_Z} > 0,$$

beta is increasing in the mean return. The market portfolio R_M has a higher expected return than the tangent portfolio $R_\wr$, $\mu_M > \mu_\wr$, there must be $\beta_\wr < \beta_M = 1$. Therefore, for an efficient portfolio moving along the frontier and away from tangent portfolio, the Sharpe ratio decreases in beta for $\beta > \beta_\wr$ when moving up, and increases for $\beta < \beta_\wr$ when moving down.

3.2 Proposition 2

FP define a BAB (betting against beta) factor as a portfolio that holds low-beta assets, leveraged to a beta of one, and that shorts high-beta assets, de-leveraged to a beta of one. Let

$$R_L = \mathbf{z}_L'\mathbf{r} \qquad R_H = \mathbf{z}_H'\mathbf{r},$$

with $\beta_L < \beta_H$, then the formula for the BAB factor is an excess return (zero-investment)

$$R_o = \frac{1}{\beta_L}(R_L - R_0) - \frac{1}{\beta_H}(R_H - R_0),$$

and the BAB's beta is zero, for $\mathrm{cov}(R_o, R_M) = 0$.

Proposition 2. *(positive expected return of BAB). The expected excess return of the zero-investment BAB factor is positive*

$$\mu_o = \frac{\beta_H - \beta_L}{\beta_H \beta_L}\psi > 0.$$

It is straightforward to compute the expected return of BAB factor R_o

$$\begin{aligned}
\mu_o = \mathrm{E}_t(R_o) &= \frac{1}{\beta_L}(\mu_L - R_0) - \frac{1}{\beta_H}(\mu_H - R_0) \\
&= \frac{1}{\beta_L}(\beta_L \mu_M + (1 - \beta_L)\,\mu_Z - R_0) - \frac{1}{\beta_H}(\beta_H \mu_M + (1 - \beta_H)\,\mu_Z - R_0) \\
&= \frac{\beta_H - \beta_L}{\beta_H \beta_L}\,(\mu_Z - R_0) = \frac{\beta_H - \beta_L}{\beta_H \beta_L}\psi > 0
\end{aligned}$$

FP assert that expected excess return of the zero-investment BAB factor is increasing in the ex ante beta spread $\frac{\beta_H - \beta_L}{\beta_H \beta_L}$ and funding tightness ψ. On funding tightness, a Lagrange multiplier is not necessary positive, hence the weighted average Lagrange multiplier ψ is not necessary positive. However, FP skip this point. Clearly, in model (2.2) with at least one constrained investor, and assume that the market portfolio has a positive excess return[4], then there is equation (2.9) to make sure $\psi > 0$.

3.3 Proposition 3

The third Proposition of FP turns out to be invalid. First, if an increase in m_k for some k does not tighten enough to bind the funding constraint, from Lemma 1, her risky investment is untouched, everything in the market remains unchanged. Second, if an increase in m_k for some k hits her funding constraint, such that

$$w_k = \frac{W_k}{m_k} \leqslant \frac{1}{\gamma_k}\left(b - aR_0\right),$$

her portfolio tilts away from the tangent portfolio, as a convex combination of all investors' portfolios, the market portfolio thus changes, so does the beta. However, in their proof for Proposition 3, FP assume no change in beta when there is a change in m_k. Mathematically, when investor k is constrained, $w_k = \frac{W_k}{m_k}$ in equation (2.5)

$$u_M = \gamma w_M = \gamma \sum_i w_i = \gamma \frac{W_k}{m_k} + \gamma \sum_{i \neq k} \min\left(\frac{W_i}{m_i}, \frac{1}{\gamma_i}\left(b - aR_0\right)\right),$$

by equation (3.2)

$$\beta = \frac{u_M}{ac - b^2 + u_M^2}\left(a\mu + \left(u_M - b\right)\right),$$

thus

$$\frac{\partial \beta}{\partial m_k} = \frac{\partial \beta}{\partial u_M}\frac{\partial u_M}{\partial m_k} = -\frac{\gamma W_k C}{m_k^2\left(ac - b^2 + u_M^2\right)^2},$$

where

$$C = \left(b - a\mu\right)u_M^2 + 2\left(ac - b^2\right)u_M + \left(ac - b^2\right)\left(a\mu - b\right),$$

unless u_M takes some special values, $C \neq 0$ and thus $\frac{\partial \beta}{\partial m_k} \neq 0$.

[4]If $R_0 < \mu_g = b/a$, where μ_g is the expected return of global minimum variance portfolio, the market portfolio has a positive excess return. However, Elton (1999) show that there are periods longer than 10 years during which stock market realized returns are on average less than the risk-free rate (1973 to 1984).

3.4 Proposition 4

Proposition 4 of FP is unfortunately also invalid. The underlying intuition for FP's equation (27) has defects: Let $\mathbf{h}_M = \sum_i \mathbf{h}_i$ be the total shares outstanding in the market, then $u_M = w_M\gamma = \gamma\mathbf{p}_t'\mathbf{h}_M$, and

$$\psi_t = \mu_{Zt} - R_0 = \frac{b - u_{Mt}}{a} - R_0 = \frac{b - \gamma\mathbf{p}_t'\mathbf{h}_M}{a} - R_0,$$

is a function of current prices $\mathbf{p}_t = [P_{1t}, \cdots, P_{st}, \cdots, P_{St}]$. However, it seems that FP are unaware of the fact that ψ_t is connected to current prices.

There is another error in their proof, in equation (28) of FP, where they incorrectly treat a_s as time unvarying. In fact, $\Omega_t = \mathrm{var}_t(P_{t+1} + \delta_{t+1}) = \mathbf{PVP}$, and thus

$$a_{st} = \mathrm{E}_t(\delta_{s,t+1}) - \gamma e_s'\Omega_t x^* = \mathrm{E}(\delta_{st}) - \gamma\mathbf{i}_s'\mathbf{PVPh}_M,$$

where $\mathbf{i}_s$ is the sth column of the identity matrix. Undoubtedly, if price is not deterministic, $\mathbf{P} = \mathrm{diag}(\mathbf{p}_t)$ should be random and a_{st} is not constant over time.

3.5 Proposition 5

Proposition 5. *(constrained investors hold high betas). Unconstrained agents hold a portfolio of risky securities that has a beta less than one; constrained agents hold portfolios of risky securities with higher betas.*

This part of FP's Proposition 5 follows immediately from Lemma 1, which states that unconstrained investors hold the tangent portfolio, and constrained investors move upward from the tangent portfolio along the efficient frontier for higher expected returns. For this reason, the market portfolio has a higher expected return than the tangent portfolio, thus unconstrained investors' portfolio beta is $\beta_\wr < \beta_M = 1$. Accordingly, constrained investors trade higher betas (risk) for higher expected returns.

FP make mistakes in the second part of their Proposition 5: "If securities s and k are identical except that s has a larger market exposure than k, $b_s > b_k$, then any constrained agent j with greater than average Lagrange multiplier, $\psi_j > \psi$, holds more shares of s than k. The reverse is true for any agent with $\psi_j > \psi$." There is a negligence in FP's equation (34), which does not guarantee that the scalar y is positive[5]. It is hard to predict who have more shares. Yet there is another misfortune in FP's equation (34), i.e., the variance matrix of residuals, Σ, is not invertible. This is so because when the market model is assumed, the variance matrix of residuals must be singular by construction. Symbolically, let

$$\mathbf{S} = \Sigma = \text{var}_t(\mathbf{e}),$$

then $\text{cov}_t(\mathbf{e}, x_{M,t+1}) = 0$ by the specification of FP's equation (13)

$$\begin{aligned} 0 &= \text{cov}_t(\mathbf{e}, x_{M,t+1}) = \text{cov}_t(\mathbf{e}, \mathbf{h}'_M \mathbf{x}_{t+1}) = \text{cov}_t(\mathbf{e}, \mathbf{x}_{t+1})\mathbf{h}_M \\ &= \text{cov}_t(\mathbf{e}, \text{E}_t(\mathbf{x}_{t+1}) + \mathbf{b}(x_{M,t+1} - \text{E}_t(x_{M,t+1})) + \mathbf{e})\mathbf{h}_M \\ &= \text{cov}_t(\mathbf{e}, \mathbf{e})\mathbf{h}_M = \mathbf{S}\mathbf{h}_M \end{aligned}$$

which shows that $\mathbf{S}$ can not be full rank.

4 Conclusions

Undoubtedly, Frazzini and Pedersen have done a great job. The main results of Frazzini and Pedersen (2014) are sound, and reload the zero-beta CAPM of Black (1972). The world for standard CAPM of Sharpe (1964) and Lintner (1965) is an ideal one, not a world to live in, while zero-beta CAPM takes a solid step towards the real world. I firmly believe that, the Z-factor, which is market neutral, is the second risk factor. The Z-factor should be the source of factor returns based on size (SMB) and book-to-market (HML) yet to be uncovered, and it is more efficient than the BAB factor by nature.

Scholars and professionals should pay more attention to the Z-factor in the zero-beta CAPM

$$\mu_A = R_0 + \beta_A(\mu_M - R_0) + (1 - \beta_A)(\mu_Z - R_0),$$

as for fund managers who are riding the beta, should be aware that the Z-factor will offset a certain amount of the expected returns.

The Z-factor may be coming back, but we are not ready for a banquet.

[5] A counterexample: let $b_1 = 2 > b_2 = 1$, and $P_1 = 1 < P_2 = 7$

$$y = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}' \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 7 \\ \frac{1}{3} \end{bmatrix} = -1 < 0$$

Appendix A Mean-Variance Framework

The mean-variance frontier has been widely known, and fully discussed in popular textbooks such as Huang and Litzenberger (1988). However, to better understand the results of FP's OLG model, I review the setup and relevant properties of the mean-variance analysis.

A.1 Pure Risky Assets

Investors trade securities $s = 1, 2, \cdots, S$ with $S > 2$, where security s pays gross return R_s with mean μ_s and variance v_s. There is a risk-free asset with gross return $R_0 > 1$, which is called risk-free rate. Let $\mathbf{r} = [R_1, R_2, \cdots, R_S]'$ be the vector of gross returns of risky assets. Assume that $\mathbf{r}$ is stationary with mean $\boldsymbol{\mu} = \mathrm{E}(\mathbf{r}) = [\mu_1, \mu_2, \cdots, \mu_S]'$ and covariance matrix $\mathbf{V} = \mathrm{var}(\mathbf{r})$ that has full rank. For convenience, define

$$a \equiv \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\,, \quad b \equiv \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{\mu}\,, \quad c \equiv \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu}\,,$$

where $\mathbf{1}$ is a conforming vector of ones.

The traditional mean-variance frontier is based on *pure risky assets* with the following assumptions:

(i) Asset prices are positive, i.e., price vector $\mathbf{p} > 0$ (every component is positive)

(ii) There are some variations in the mean return, say $\boldsymbol{\mu} \neq l\mathbf{1}$, for any real number l

(iii) There are no redundant securities, i.e., $\mathbf{V} > 0$ (positive definite)

For pure risky assets, we have $a > 0, c > 0$ and $ac - b^2 > 0$. Define

$$h(x) = ax^2 - 2bx + c,$$

then

$$h(R_0) = aR_0^2 - 2bR_0 + c = (\boldsymbol{\mu} - R_0\mathbf{1})'\,\mathbf{V}^{-1}\,(\boldsymbol{\mu} - R_0\mathbf{1}) > 0.$$

A.2 Mean-Variance Frontier

Now we are ready to review the properties of the mean-variance frontier. For portfolio of pure risky assets, $R = \mathbf{z}'\mathbf{r}$ is on the mean-variance frontier F if and only if

$$v = \mathrm{var}(R) = \frac{h(\mu)}{ac - b^2}, \tag{A.1}$$

where $\mu = \mathrm{E}(R)$, and the portfolio weights are

$$\mathbf{z} = \frac{a\mu - b}{ac - b^2}\mathbf{V}^{-1}\boldsymbol{\mu} + \frac{c - b\mu}{ac - b^2}\mathbf{V}^{-1}\mathbf{1}.$$

For the global minimum variance portfolio (MVP) $R_g \in \mathsf{F}$

$$\mu_g = \frac{b}{a} \qquad v_g = \frac{1}{a} \qquad \mathbf{z}_g = \frac{1}{a}\mathbf{V}^{-1}\mathbf{1}.$$

A.3 Zero-Beta Portfolio

Let $R_p \in \mathsf{F}$ be a frontier portfolio, and $R_p \neq R_g$, then there exists a unique zero-beta (zero-covariance) portfolio $R_z \in \mathsf{F}$ such that $\mathrm{cov}(R_p, R_z) = 0$, with the expected return

$$\mu_z = \frac{b\mu_p - c}{a\mu_p - b}.$$

For any asset or portfolio R_A (may or may not on F), we have

$$\mu_A = \mu_z + \beta_{Ap}(\mu_p - \mu_z), \tag{A.2}$$

where

$$\beta_{Ap} = \frac{\mathrm{cov}(R_A, R_p)}{\mathrm{var}(R_p)}.$$

A.4 The Tangent Portfolio

Considering the inclusion of the risk-free asset, $R = z_0 R_0 + \mathbf{z}'\mathbf{r}$ is on the mean-variance frontier F_0 if and only if portfolio weights

$$z_0 = \frac{(aR_0 - b)\,\mu + c - bR_0}{h\,(R_0)},$$
$$\mathbf{z} = \frac{\mu - R_0}{h\,(R_0)}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - R_0\mathbf{1}\right).$$

Assuming $R_0 < \mu_g = b/a$, the so-called tangent portfolio $R_\wr$ (on both F and F_0) has weights $z_{\wr 0} = 0$ and

$$\mathbf{z}_\wr = \frac{1}{b - aR_0}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - R_0\mathbf{1}\right),$$

and expected return

$$\mu_\wr = \frac{c - bR_0}{b - aR_0}.$$

Coincidently, the expected return of tangent portfolio $R_\wr$'s zero-beta portfolio equals the risk-free rate R_0.

A.5 Zero-Beta CAPM

In equation (A.2), choosing R_p as the tangent portfolio $R_\wr$ will produce the standard CAPM of Sharpe (1964) and Lintner (1965) (where the market portfolio R_M equals the tangent portfolio $R_\wr$)

$$\mu_A = R_0 + \beta_A(\mu_M - R_0),$$

with

$$\beta_A = \frac{\text{cov}(R_A, R_M)}{\text{var}(R_M)}.$$

When there are heterogeneous investors, the market portfolio may be still on the frontier, but would not coincide with the tangent portfolio, namely $R_M \neq R_\wr$. In this case, equation (A.2) becomes zero-beta CAPM (Black, 1972)

$$\mu_A = \mu_Z + \beta_A(\mu_M - \mu_Z), \tag{A.3}$$

where μ_Z is the expected return of $R_Z \in \mathsf{F}$, the zero-beta portfolio of market portfolio.

References

Black, Fischer, 1972. Capital Market Equilibrium with Restricted Borrowing. *Journal of Business*, 45(3):444–455

Black, Fischer, 1993. Beta and Return. *Journal of Portfolio Management*, 20(1):8–18

Elton, Edwin J., 1999. Expected Return, Realized Return, and Asset Pricing Tests. *Journal of Finance*, 54(4):1199–1220

Frazzini, Andrea and Lasse Heje Pedersen, 2014. Betting against Beta. *Journal of Financial Economics*, 111(1):1–25

Huang, Chi-fu and Robert H. Litzenberger, 1988. *Foundations for Financial Economics*. Prentice Hall, Englewood Cliffs, NJ

Lintner, John, 1965. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, 47(1):13–37

Sharpe, William F., 1964. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance*, 19(3):425–442

Sharpe, William F., 1970. *Portfolio Theory and Capital Markets*. McGraw-Hill Professional, New York