Estimating Liquidity Premium of Corporate Bonds Using the Spread Information in On- and Off-the-Run Treasury Securities

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Abstract

Liquidity risk has long been thought to be an important factor of bond pricing. However, measuring and tracking liquidity spreads remains an elusive task. One of the major obstacles is that liquidity risk is often confounded with effects of other factors which are difficult to disentangle empirically. In this paper we employ a reduced-form approach to model the liquidity spread component of off-the-run Treasuries. Using the liquidity intensity estimated from the spread of off-the-run issues as a state variable, we jointly estimate the components of default and liquidity spreads from corporate bond prices. We find that liquidity spread is highly correlated with traditional liquidity measures such as bid-ask spread, volume, order imbalance and depth. Liquidity spreads of corporate bonds increase with maturity and credit risk. On average liquidity spread is close to 30 percent of the spread of corporate bonds, 25 percent of the spread for investment-grade bonds and 35 percent for speculative-grade bonds. The results suggest that liquidity risk is an important determinant of corporate bond spreads.

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Liquidity risk has been thought to be an important factor affecting bond pricing. However, measuring and tracking liquidity spreads remains an elusive task. One of the major obstacles is that liquidity risk is often confounded with effects of other factors (e.g., default, information and market risks), which are difficult to disentangle empirically (see Duffie and Singleton, 1999; Duffee, 1999). Liquidity is also a broad concept, which may be referred to as ease of accessing funds or trading assets, or a state factor that systematically affects asset pricing. Different concepts could lead to very different liquidity metrics. Unless it is properly defined, measuring and comparing liquidity effects can be a very challenging task. Liquidity in bond markets has received a great deal of attention recently. This heightened concern is attributable to the episode of Long-term Capital Management in the fall of 1998, which precipitated a widespread deterioration in liquidity across markets.¹

In this paper, we focus on the aspect of liquidity associated with the pricing of bonds. We propose a new method to resolve the problem of identifying the liquidity spread of corporate bonds. The proposed model is tractable and easy to implement. The implementation of the model involves two steps. First, we employ a reduced-form approach to model the liquidity intensity process of off-the-run Treasury securities. This permits us to extract important liquidity information from the U.S. Treasury market. Using the liquidity factor estimated from the spread of off-the-run issues as a state

¹ See Duffie and Ziegler (2001) and Pastor and Stambaugh (2003). There have also been a concern about liquidity arising from the federal government’s reduced funding needs since then and the resultant reduction in the supply of Treasuries (see Fleming, 2003).
variable in the term structure of defaultable bonds, we then jointly estimate the parameters of default and liquidity intensities of corporate bonds.

On- and off-the-run Treasury securities are ideal instruments for extracting the information for illiquidity. First, unlike corporate or municipal bonds, these Treasury securities are riskfree and so differences in liquidity risk cannot be confounded with credit risk. Second, given maturity, these securities should have the same interest rate risk or market risk. Hence, there is no need to assume any asset pricing model to control for the systematic risk effect when comparing the prices of two Treasury securities with identical maturities. Third, to the extent that Treasury securities are affected by the same information, differences in their pricing are less likely to be caused by the information factor. Thus, using Treasury security data allows us to construct a more reliable measure of market-wide liquidity, which is free from the effects of both systematic and idiosyncratic risk factors.

Existing term structure models of defaultable bonds are unable to explain corporate yield spreads satisfactorily. A consensus is that there are missing factors in the traditional term structure models and a major suspect is liquidity risk. Modeling liquidity risk of corporate bonds is complicated because it is inherently difficult to separate liquidity from default premiums (see Duffie and Singleton, 1999). To overcome this difficulty, we employ the liquidity intensity extracted from the off-the-run Treasury yields as a state factor in the term structure of corporate straight bonds. We then formulate the liquidity intensity process of defaultable bonds as a function of this

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2 One potential issue is the incomplete information problem caused by noisy accounting information as indicated by Duffie and Lando (2001). This is lesser concern for government bonds.
liquidity state factor. This setup helps capture in a tractable way the response of corporate bonds to the market-wide liquidity condition.

Our empirical formulation of the liquidity intensity process of corporate bonds is quite appropriate because liquidity in Treasury markets affects the pricing of virtually every other asset in financial markets.\(^3\) This spillover relationship is undoubtedly more intimate between corporate bond and Treasury markets (see Longstaff, 2004), as evidenced by the empirical finding that the liquidity component of corporate bond spreads is strongly related to the on- and off-the-run spread (see Longstaff, Neis and Mithal, 2004). Thus, including the liquidity factor embedded in the spread of Treasury issues in the term structure model allows us to capture very effectively the systematic variation in corporate bond yields to bond market liquidity. More importantly, using the information for Treasury market liquidity as an instrumental variable resolves an identification problem in empirical estimation when attempting to separate the effect of liquidity risk from that of default risk on corporate bond spreads.

Several papers are related to our study. Amihud and Mendelson (1991) examine the effect of the liquidity of Treasury notes relative to bills over a short maturity (less than 6 months). They find that less liquid Treasury notes have higher yields than Treasury bills. Unlike their study, we model liquidity of off-the-run Treasury securities using a reduced-form approach (Jarrow and Turnbull, 1995; Duffie and Singleton, 1999) and examine the effect of liquidity risk on bond pricing. Elton and Green (1998) examine the effects of taxes and liquidity and find that both effects on Treasury bond prices are small. Jarrow (2001) models liquidity risk using a convenience yield approach where liquidity premium is analogous to a convenience yield from holding an illiquid bond in
the investor’s portfolio.\textsuperscript{4} Goldreich, Hanke and Nath (2002) examine the yield difference of on-the-run and off-the-run issues from the perspective of future liquidity. de Jong and Driessen (2004) show that liquidity is a priced factor in an APT model for expected returns and the exposures of corporate bond returns to fluctuations in Treasury bond liquidity and equity market liquidity help explain the credit spread puzzle. Brandt and Kavajecz (2004) find that orderflow imbalances in the presence of an illiquid market have a greater impact on Treasury yields than orderflow imbalances in a liquid market. Longstaff, Neis and Mithal (2004) utilize the information in credit-default swaps to measure the size of the default and nondefault components in corporate spreads. Similar to their study, we employ the reduced-form approach to model spread components. Unlike their model, we use the information in the spread of off- and on-the-run Treasuries as a liquidity risk factor to estimate the liquidity component of corporate spreads. In the Longstaff et al. study, the nondefault component contains not only the liquidity premium but the potential effects of other factors.\textsuperscript{5} In contrast, we directly estimate the size of liquidity premium from the reduced-form model. Our study thus complements their work using credit-default swaps data.

We fit the price of off-the-run Treasury issues by a three-factor affine model including two interest rate factors and a liquidity intensity process. This model captures the rich dynamics of off-the-run Treasury yield curves and time-varying liquidity spreads, and thus fits the data extremely well. Liquidity risk premiums of off-the-run Treasury securities are significant in economic terms. The estimated liquidity spread ranges from 8 to 19 basis points, with its magnitude increasing with time to maturity, and the length of

\textsuperscript{3} See Brandt and Kavajecz (2004).
\textsuperscript{4} See also Janosi, Jarrow and Yildirim (2002).
time that an issue is off the run. We also find significant time variations in the liquidity premiums embedded in Treasury yields. The variation in liquidity has apparently increased quite substantially in recent years, suggesting that liquidity risk has heightened lately. This trend might have an adverse effect on security prices by raising the level of liquidity premium for the financial market. Furthermore, estimates of off-the-run liquidity spread are highly correlated with traditional liquidity measures such as bid-ask spread, volume, order imbalance and depth. This finding suggests that the extracted liquidity factor contains important market-wide liquidity information.

The pricing of corporate bonds is fitted by a four-factor affined model including a default intensity process and a liquidity intensity process linked to the off-the-run liquidity intensity. We find that liquidity spreads of individual corporate bonds vary considerably (10 to 49 percent of total spread) cross-sectionally, and increase with maturity and credit risk of corporate bonds. The liquidity spread is sizable which on average is close 30 percent of corporate bond spreads. Liquidity spread accounts for about 25 percent of total spread for investment-grade bonds, and 35 percent for speculative-grade bonds. The results suggest that liquidity risk is an important determinant of corporate bond yield spreads.

The remainder of the paper is organized as follows. Section 1 presents the models for estimating the liquidity spreads of off-the-run Treasuries and corporate bonds. Section 2 discusses empirical methodology and the estimation procedure. Section 3 describes data and presents empirical results. Finally, Section 4 concludes the paper.

\footnote{However, they show that most of the nondefault component is attributed to liquidity.}
1. Modeling Liquidity Spreads of Default-Free and Defaultable Bonds

1.1 The empirical processes

Denote the time \( t \) price of zero-coupon on-the-run issues that pays off a dollar at time \( T \) as \( V_{on}(t,T,0) \). Then under the standard no-arbitrage assumption, the price of this on-the-run Treasury issue is

\[
V_{on}(t,T,0) = E^Q_t \exp \left[ - \int_t^T r_u du \right]
\]  

where \( r_i \) is the instantaneous risk-free interest rate and the expectation is taken with respect to the risk-neutral (equivalent martingale) measure \( Q \).

Following Duffee (1999), we adopt a two-factor term structure model where \( r_t \) equals the sum of a constant and the two factors, \( s_1 \) and \( s_2 \)

\[
r_t = \alpha_r + s_{1,t} + s_{2,t}
\]  

The conventional interpretation is that the first factor represents the slope of the Treasury yield curve and the second reflects the level of Treasury yields (see Pearson and Sun, 1994).

The dynamics of the two interest rate factors are assumed to follow a square-root process

\[
ds_{i,t} = \kappa_{si} (\theta_{si} - s_{i,t}) dt + \sigma_{si} \sqrt{s_{i,t}} dZ_{i,t} \quad \text{for} \quad i = 1,2
\]  

where \( \kappa_{si}, \theta_{si}, \sigma_{si} \) are constants and \( dZ_{1,t} \), \( dZ_{2,t} \) are independent Brownian motions.

Under the equivalent martingale measure, these processes can be expressed as

\[
ds_{i,t} = (\kappa_{si} \theta_{si} - (\kappa_{si} + \eta_{si}) s_{i,t}) dt + \sigma_{si} \sqrt{s_{i,t}} d\hat{Z}_{i,t} \quad \text{for} \quad i = 1,2
\]
where \( d\hat{Z}_{i,t} \), \( d\hat{Z}_{2,t} \) are independent Brownian motions and \( \eta_{si} \) is the market price of risk associated with \( d\hat{Z}_{i,t} \).

The main difference between off- and on-the-run issues with the same maturity and coupon is liquidity. Because an off-the-run issue is less liquid, it is expected to offer a higher yield. We model liquidity as a stochastic intensity process \( \zeta \). Given this liquidity intensity process, the price of a zero-coupon off-the-run bond can be written as

\[
V_{off}(t,T,0) = E^Q_t \exp \left[ -\int_t^T \left( r_u + \zeta_u \right) \, du \right]
\]  

(5)

We characterize liquidity of the off-the-run issue by a single-factor square-root process plus two components linked to interest rate factors. Specifically, liquidity of the off-the-run issue has the following translated square-root process:

\[
\zeta_t = \alpha_i + l_t + \beta_1(s_{1,t} - \overline{s}_{1,t}) + \beta_2(s_{2,t} - \overline{s}_{2,t})
\]  

(6)

The higher the intensity, the less liquid the off-the-run issue. We postulate that liquidity intensity is negatively related to interest rate factors \( \beta_1, \beta_2 < 0 \). The stochastic component of the liquidity intensity process \( l_t \) unrelated to interest rates is assumed to follow a single factor square-root process

\[
dl_t = \kappa_i (\theta_i - l_t) \, dt + \sigma_i \sqrt{l_t} \, d\hat{Z}_i
\]  

(7)

Under the equivalent martingale measure, the stochastic process \( l_t \) can be expressed as

\[
dl_t = (\kappa_i \theta_i - (\kappa_i + \eta_i)l_t) \, dt + \sigma_i \sqrt{l_t} \, d\hat{Z}_i
\]  

(8)

where \( \eta_i \) is the market price of risk associated with \( d\hat{Z}_i \).

1.2 Pricing formulas for on- and off-the-run issues

Given the stochastic processes of interest rate factors, we can obtain the pricing
formula of zero-coupon on-the-run issues, which has the following form:

\[ V_{on}(t, T, 0) = \exp\left[ \psi_0(t) - s_{1,t} \psi_1(t) - s_{2,t} \psi_2(t) \right] \] 

(9)

Assuming that the zero-coupon bond pays one dollar at maturity, the terminal condition is

\[ V_{on}(T, T, 0) = 1 \quad \forall s_1, s_2. \] 

The solution to (9) is given by

\[ \psi_0(t) = -A(T-t) + \sum_{i=1}^{2} \psi_{0,i} \] 

(10)

\[ \psi_{0,i} = \frac{2\kappa_{si} \theta_{si}}{\sigma_{si}^2} \log \left[ \frac{2\gamma_i e^{\frac{1}{2} (\kappa_{si} + \eta_{si}) (T-t)}}{2\gamma_i + (\kappa_{si} + \eta_{si} + \gamma_i)(e^{\gamma_i(T-t)} - 1)} \right], \quad i = 1, 2 \] 

(11)

and

\[ \psi_i(t) = \frac{2(\gamma_i(T-t) - 1)}{2\gamma_i + (\kappa_{si} + \eta_{si} + \gamma_i)(e^{\gamma_i(T-t)} - 1)} \quad i = 1, 2 \] 

(12)

where \( \gamma_i = \sqrt{(\kappa_{si} + \eta_{si})^2 + 2\sigma_{si}^2}. \)

Similarly, we can obtain the pricing formula for the off-the-run issue. The closed-form solution for the off-the-run issue is given by

\[ V_{off}(t, T, 0) = \exp\left[ \psi_0(t) - s_{1,t}^* \psi_1(t) - s_{2,t}^* \psi_2(t) - l \psi_f(t) \right] \] 

(13)

where

\[ s_{1,u} = [1 + \beta_1] s_{1,u}, \quad s_{2,u} = [1 + \beta_2] s_{2,u} \]

\[ \psi_0(t) = -A(T-t) + \sum_{i=1}^{2} \psi_{0,i} + \psi_{0,f} \] 

(14)

\[ \psi_{0,i} = \frac{2\kappa_{si} \theta_{si}^*}{\sigma_{si}^*} \log \left[ \frac{2\gamma_i e^{\frac{1}{2} (\kappa_{si} + \eta_{si} + \gamma_i)(T-t)}}{2\gamma_i + (\kappa_{si} + \eta_{si} + \gamma_i)(e^{\gamma_i(T-t)} - 1)} \right] \] 

(15)
\[
\psi_\omega(t) = \frac{2\kappa_i^* \theta_i}{\sigma_i^*} \log \left[ \frac{2\gamma_i e^{\gamma_i/t}}{2\gamma_i + (\kappa_i + \eta_i + \gamma_i)(e^{\gamma_i/t} - 1)} \right]
\]

\[
\psi_\delta(t) = \frac{2(e^{\gamma_i/t} - 1)}{2\gamma_i + (\kappa_i + \eta_i + \gamma_i)(e^{\gamma_i/t} - 1)}
\]

\[
\gamma_i = \sqrt{(\kappa_i + \eta_i)^2 + 2\sigma_i^2}
\]

\[
\theta_i^* = \theta_i (1 + \beta_i)
\]

\[
\sigma_i^* = \sigma_i \sqrt{1 + \beta_i}, \quad i = 1, 2
\]

### 1.3 Default and liquidity premia in corporate bond pricing

We next turn to the pricing model of corporate bonds. Consider a zero-coupon straight (non-callable) corporate bond that pays a dollar at maturity if the firm does not default by maturity \( T \). Conversely, if the firm defaults at or before maturity, investors will receive only a fraction of the market value of bond, \( \delta \), where \( 0 \leq \delta < 1 \). The time \( t \) price of the zero-coupon defaultable bond with maturity \( T \) and recovery rate of \( \delta \) is given by

\[
V_d(t, T, 0, \delta) = E_t^Q \exp \left[ -\int_t^T \left( r_u + \zeta_u^* + (1 - \delta) h_u \right) du \right]
\]

where \( \zeta^* \) and \( h \) are intensities of liquidity and default for the corporate bond, respectively.

Financial markets are correlated in the sense that an unusual movement in one market can have powerful impacts on other markets of different instruments and
structures (see Collin-Dufresne, Goldstein and Helwege, 2003). That liquidity problem in one market can spread quickly to others is evidenced by the event of Long-term Capital Management. This cross-market liquidity correlation is strong between Treasury and corporate bond markets as exhibited in the close relationship between the liquidity component of corporate bond spread and the on-the-run/off-the-run spread documented by previous studies.\(^6\) We capture this intimate correlation by postulating that the liquidity intensity of corporate bonds is a proportional function of the liquidity intensity \(\zeta\) of off-the-run Treasuries; that is, \(\zeta' = \omega \zeta\), where \(\omega\) is a sensitivity parameter. From (6), we can rewrite the liquidity intensity process of corporate bonds as
\[
\zeta_i' = \omega(\alpha_i + l_i + \beta_1(s_{1,i} - \bar{s}_{1,j}) + \beta_2(s_{2,i} - \bar{s}_{2,j})) \\
= a + \omega l_i + b_1(s_{1,i} - \bar{s}_{1,j}) + b_2(s_{2,i} - \bar{s}_{2,j})
\]
In this formulation, \(\omega l_i\) is the component of the corporate bond liquidity spread linked to the stochastic liquidity condition of bond market, \(b_1(s_{1,i} - \bar{s}_{1,j}) + b_2(s_{2,i} - \bar{s}_{2,j})\) is the component related to the interest rate factors and \(a\) is the component idiosyncratic to the individual corporate bond. The part of the liquidity intensity unrelated to interest rate changes is \(l_i' = a + l_i'\), where \(l_i' = \omega l_i\).

We use the liquidity intensity of off-the-run issues, instead of the observed on-and off-the-run spread, as a state factor in the liquidity process of corporate bonds for the following reasons. First, the measure of yield spread of on-the-run/off-the-run may be biased by stale prices of off-the-run issues, which are often infrequently traded, especially for those issues which have long maturity and have been off the run for quite some time.

\(^6\) Longstaff (2004, p. 511) also indicates that credit crunches and sudden changes in the corporate bond market condition has a direct significant impact on the U.S. Treasury market.
Empirical evidence has shown on- and off-the-run spreads are sometimes negative (see Brandt and Kavajecz, 2004) due to the inactive trading of off-the-run issues.\textsuperscript{7} Using the liquidity intensity estimated from the term structure model of off-the-run issues avoids this negative spread problem. Second, in estimating the liquidity intensity of individual corporate bonds, the time series of the liquidity factor must match that of the corporate bond data. Since the liquidity intensity of off-the-run issues is estimated from the model, we can easily generate direct estimates of the liquidity factor implied by the on-the-run/off-the-run spread for any length of time between trades to match the time-series of corporate bond prices.

The liquidity risk of corporate bonds should be higher than that of Treasuries because corporate bonds are less frequently traded and have lower trading volume. Thus, when there is a market-wide liquidity problem, its impact on corporate bonds is expected to be greater than on Treasuries ($\omega > 1$). In addition, since lower-grade bonds tend to have higher liquidity risk due to their lower marketability, $\omega$ should increase with credit risk. Substituting the function of corporate bond liquidity intensity $\zeta^\prime$ into (19), we can express the zero-coupon corporate bond price as

$$V_d(t, T, 0, \delta) = E^0_t \exp \left[ -\int_t^T \left( r_u + \omega \zeta_u + (1 - \delta) h_u \right) du \right]$$

(20)

The zero-coupon straight bond price is given by the expectation of the cumulative adjusted discount rate $r_t + \zeta_t + (1 - \delta) h_t$ over the period of $[t, T]$ under the risk-neutral measure.

The default intensity process is characterized by a translated single-factor square-root process with two interest rate factors:

\textsuperscript{7} See Brandt and Kavajecz (2004, p. 2630).
\[ h_t = \frac{1}{1-\delta} \left[ \alpha_d + h_t^* + \beta_{d1}(s_{1,t} - \overline{s}_{1,t}) + \beta_{d2}(s_{2,t} - \overline{s}_{2,t}) \right] \]  

(21)

It contains a constant term \( \alpha_d \), a stochastic process \( h_t^* \), and two interest-rate sensitivity components \( \beta_{d1}(s_{1,t} - \overline{s}_{1,t}) + \beta_{d2}(s_{2,t} - \overline{s}_{2,t}) \), which captures the relationship between corporate bond yields and variations in the default-free term structure (see Duffee, 1999).

The default intensity due to the firm’s idiosyncratic factor is formulated as

\[ dh_t^* = \kappa_d(\theta_d - h_t^*)dt + \sigma_d \sqrt{h_t^*} dZ_t^d \]  

(22)

Under the equivalent martingale measure, the stochastic process \( h_{j,t}^* \), can be expressed as

\[ dh_t^* = (\kappa_d \theta_d - (\kappa_d + \eta_d)h_t^*)dt + \sigma_d \sqrt{h_t^*} d\hat{Z}_t^d \]  

(23)

The adjusted discount rate can be rewritten as

\[
\begin{align*}
    r_t + \hat{\zeta}_t + (1-\delta)h_t = & \quad \alpha_r + s_{1,t} + s_{2,t} \\
    & + \omega[\alpha_i + l_t + \beta_1(s_{1,t} - \overline{s}_{1,t}) + \beta_2(s_{2,t} - \overline{s}_{2,t})] \\
    & + [\alpha_d + h_t^* + \beta_{d1}(s_{1,t} - \overline{s}_{1,t}) + \beta_{d2}(s_{2,t} - \overline{s}_{2,t})] \\
    = & \quad A_d + s_{1,t}^* + s_{2,t}^* + l_t + h_t^*
\end{align*}
\]

(24)

where \( A_d = \alpha_r + \omega \alpha_i + \alpha_d - \omega \beta_1 \overline{s}_{1,t} - \omega \beta_2 \overline{s}_{2,t} - \beta_{d1} \overline{s}_{1,t} - \beta_{d2} \overline{s}_{2,t} \), \( s_{1,t}^* = [1 + \omega \beta_1 + \beta_{d1}]s_{1,t} \), and \( s_{2,t}^* = [1 + \omega \beta_2 + \beta_{d2}]s_{2,t} \). The dynamics of the two adjusted interest rate factors are

\[
\begin{align*}
    ds_{i,t}^* = & \quad \kappa_{si}(\theta_{si} - s_{i,t}^*)dt + \sigma_{si}^* \sqrt{s_{i,t}^*} dZ_{i,t}^s \\
    \theta_{si}^* = & \quad \theta_{si} (1 + \beta_{d1} + \omega \beta_i) \\
    \sigma_{si}^* = & \quad \sigma_{si} \sqrt{1 + \beta_{d1} + \omega \beta_i} \\
\end{align*}
\]

(25)

where \( dZ_{1,t}^s \) and \( dZ_{2,t}^s \) are independent. Under the equivalent martingale measure, these processes are represented by

\[
\begin{align*}
    ds_{i,t}^* = & \quad (\kappa_{si}\theta_{si}^* - (\kappa_{si} + \eta_{si})s_{i,t}^*)dt + \sigma_{si}^* \sqrt{s_{i,t}^*} d\hat{Z}_{i,t}^s, \\
\end{align*}
\]

(26)

where \( d\hat{Z}_{1,t}^s \) and \( d\hat{Z}_{2,t}^s \) remain independent.
Substituting the expressions for the spot rate and the intensities into the pricing function and using the independence condition for Brownian motions yields

\[
V_d(t, T, 0, \delta) = \exp\left[-A_d(T-t)\right]E_t^Q \exp\left[-\int_t^T s_{1,u}^* du\right] E_t^Q \exp\left[-\int_t^T s_{2,u}^* du\right] \times
E_t^Q \exp\left[-\int_t^T \ell'_u du\right] E_t^Q \exp\left[-\int_t^T h'^*_u du\right]
\]  \hspace{1cm} (27)

We seek a solution of the following form:

\[
V_d(t, T, 0, \delta) = \exp\left[\psi_{d0}(t) - s_{1,t}^* \psi_1(t) - s_{2,t}^* \psi_2(t) - \ell'_t \psi_l(t) - h'^*_t \psi_h(t)\right]
\]  \hspace{1cm} (28)

At maturity, we have

\[
V_c(T, T, 0, \delta, k) = 1 \quad \forall s_{1,t}^*, s_{2,t}^*, \ell, h'^*_t
\]

The initial conditions are

\[
\psi_1(T) = \psi_2(T) = \psi_d(T) = \psi_c(T) = \psi_{d0}(T) = 0
\]

The closed-form solution to (28) is

\[
\psi_i(t) = \frac{2\left(e^{\gamma_i(T-t)} - 1\right)}{2\gamma_i + (\kappa_{si} + \eta_{si} + \gamma_i)(e^{\gamma_i(T-t)} - 1)} \quad i = 1, 2, \quad \gamma_i = \sqrt{(\kappa_{si} + \eta_{si})^2 + 2\sigma_{si}^2}
\]  \hspace{1cm} (29)

\[
\psi_j(t) = \frac{2\left(e^{\gamma_j(T-t)} - 1\right)}{2\gamma_j + (\kappa_{sj} + \eta_{sj} + \gamma_j)(e^{\gamma_j(T-t)} - 1)} \quad j = l, d, \quad \gamma_j = \sqrt{(\kappa_{sj} + \eta_{sj})^2 + 2\sigma_{sj}^2}
\]  \hspace{1cm} (30)

and

\[
\psi_{d0}(t) = -A_d(T-t) + \psi_{01} + \psi_{02} + \psi_{0l} + \psi_{0d}
\]  \hspace{1cm} (31)

where

\[
\psi_{0i}(t) = \frac{2\kappa_{si}\theta_{si}^*}{\sigma_{si}^2} \log\left[\frac{2\gamma_i e^{\gamma_i(T-t)}}{2\gamma_i + (\kappa_{si} + \eta_{si} + \gamma_i)(e^{\gamma_i(T-t)} - 1)}\right] \quad i = 1, 2
\]  \hspace{1cm} (32)
\[
\psi_{0j}(t) = \frac{2\kappa_j \theta_j}{\sigma_j^2} \log \left[ \frac{2\gamma_j e^{\frac{1}{2}(\kappa_j + \eta_j + \gamma_j)(T-t)}}{2\gamma_j + (\kappa_j + \eta_j + \gamma_j)(e^{\gamma_j(T-t)} - 1)} \right] \quad j = l, d \quad (33)
\]

The state variables for the interest rate process and the liquidity intensity process \( \zeta \) are estimated from the term structures of on- and off-the-run Treasury issues. Substituting these estimates into (28), we obtain the estimate for the zero-coupon bond price:

\[
V_d(t, T, 0, \delta) = \exp \left[ \psi_{d0}(t) - \tilde{s}_{1l} \psi_1(t) - \tilde{s}_{2l} \psi_2(t) - \tilde{h}_l \psi_1(t) - h_l \psi_d(t) \right] \quad (34)
\]

The yield spread estimate is

\[
YSP_{straight} = \frac{-\psi_{d0}(t) + \tilde{s}_{1l} \psi_1(t) + \tilde{s}_{2l} \psi_2(t) + \tilde{h}_l \psi_1(t) + h_l \psi_d(t)}{T - t} \quad (35)
\]

Default and liquidity premiums are then estimated as \( DS = YSP_{straight} \mid_{\zeta = 0} \) and \( YSP_{straight} \mid_{h_l = 0} \), respectively.

A coupon bond can be treated as a combination of zero-coupon bonds. Thus, the price of coupon bond is

\[
P_d(t, T) = \sum_{i=1}^{T} CV_d(i, T, 0) + V_d(T, T, 0)
\]

\[
= \sum_{i=1}^{T} Ce^{\left[\psi_{d0}(i) - \tilde{s}_{1l} \psi_1(i) - \tilde{s}_{2l} \psi_2(i) - \tilde{h}_l \psi_1(i) - h_l \psi_d(i)\right]}
\]

\[
+ Ce^{\left[\psi_{d0}(T) - \tilde{s}_{1l} \psi_1(T) - \tilde{s}_{2l} \psi_2(T) - \tilde{h}_l \psi_1(T) - h_l \psi_d(T)\right]} \quad (34a)
\]

where \( P_d(t, T) \) is the time \( t \) price for a corporate bond with maturity \( T \) and coupon rate \( C \).
2. Empirical Methodology

We employ the Kalman filter method (Duffee, 1999; Chen and Scott, 2003) to estimate the term structure models of Treasury issues. The exact transition density of the state variables is non-central $\chi^2$. We use the quasi-maximum-likelihood (QML) method for Kalman filter estimation where the non-central $\chi^2$ transition density is substituted by normal density. The estimation procedure involves two steps. In step one, we estimate the parameters of the interest rate process of on-the-run Treasuries. In step two, the estimated interest rate factors are incorporated into the off-the-run Treasury term structure. For corporate bonds both the interest rate factors and liquidity factor estimated in step one are incorporated into the term structure to estimate the parameters of the model.

2.1 On-the-run Treasuries

Most Treasury issues bear coupons. The time $t$ price of on-the-run coupon bonds with maturity $T$ and coupon $C$ can be written as

$$P_{on}(t,T) = f(S_i) = \sum_{i=t}^{T} CV_{on}(i,T,0;S_i) + V_{on}(T,T,0;S_i)$$

(36)

The measurement and transition equations of on-the-run Treasury prices are

$$P_{on} = f(S_i) + \varepsilon_t$$
$$E_{t-1}(\varepsilon_t \varepsilon_t') = U_t$$

(37)

$$S_t = d + \phi S_{t-1} + \eta_t$$
$$E_{t-1}(\eta_t \eta_t') = Q_t$$

(38)

where the state variable vector $S_t = (s_{1,t}, s_{2,t})'$ and $U_t$ and $Q_t(S_{t-1})$ are diagonal matrices corresponding to the variances of the measurement errors of prices and state variables, respectively. The transitional equation components $d$ and $\phi$ are
\[ d = \begin{pmatrix} \theta_{s1}(1-e^{-x_{s1}/12}) \\ \theta_{s2}(1-e^{-x_{s2}/12}) \end{pmatrix}, \quad \phi = \begin{pmatrix} e^{-x_{s1}/12} & 0 \\ 0 & e^{-x_{s2}/12} \end{pmatrix} \]  

(39)

and \( Q_i(S_{t-1}) \) is a 2 x 2 diagonal matrix with elements

\[ Q_i(S_{t-1}) = \kappa_{si}^{-1} \sigma_{si}^2 [s^1_{i,t-1}(e^{-x_{si}/12} - e^{-2x_{si}/12}) + (\theta_{si}/2)(1-e^{-x_{si}/12})^2] \text{ for } i = 1, 2. \]  

(40)

In (37), the coupon bond price is a nonlinear function of the state variable vector \( S_t = (s_{1,t}, s_{2,t})' \). We linearize the function \( f(S_t) \) around the one-period forecast of \( S_t \), \( S_{t|t-1} = (s_{1,t|t-1}, s_{2,t|t-1})' \), and evaluate \( Q_i(S_{t-1}) \) at the contemporaneous prediction of \( S_{t-1} \). The procedure of linearization is explained in the Appendix. After linearization, the measurement equation becomes

\[ P_{on}(t,T) = Z_t + Z_t(S_t - S_{t|t-1}) + \varepsilon_t, \quad E_{t-1}(\varepsilon_t, \varepsilon'_t) = U_t \]  

(41)

with the transition equation

\[ S_t = d + \phi S_{t-1} + \eta_t, \quad E_{t-1}(\eta_t, \eta'_t) = Q_t \]  

(42)

where \( Z_0 \) and \( Z_t \) are as defined in (A4)-(A6) of the Appendix.

2.2 Off-the-run Treasuries

To estimate the liquidity intensity process, we input the estimates of the two state variables (interest rate factors) from the on-the-run model into the pricing model of the off-the-run coupon issue, \( P_{off}(t,T) = \sum_{i=t}^{T} CV_{off}(i,T,0) + V_{off}(T,T,0) \). The measurement and transition equations of the off-the-run issue are

\[ P_{off} = g(l_t, \hat{S}_t) + \varepsilon_{i,t}, \quad E_{t-1}(\varepsilon_{i,t}, \varepsilon'_{i,t}) = U_{i,t} \]  

(43)

\[ l_t = d_i + \phi l_{t-1} + \eta_{i,t}, \quad E_{t-1}(\eta_{i,t}, \eta'_{i,t}) = Q_{i,t} \]  

(44)

\[ A \text{ similar linearization procedure is adopted by Duffee (1999).} \]
where

\[
d_t = \left[ \theta_t \left(1 - e^{-\kappa_t/12}\right) \right], \quad \phi_t = e^{-\kappa_t/12}
\]  \tag{45}

\[
Q_t \left( l_{t-\tau}, \tau \right) = \kappa_t^{-1}\sigma_t^2 \left[ l_{t-\tau} \left( e^{-\kappa_t/12} - e^{-\kappa_t/6} \right) + \frac{\theta_t}{2} \left(1 - e^{-\kappa_t/12}\right)^2 \right]
\]  \tag{46}

The function \( g(l_t, \hat{S}_t) \) maps the liquidity intensity into the price of off-the-run issues where \( \hat{S}_t \) corresponds to the vector of the predicted values of state variables. Off-the-run issues are less frequently traded and there may not be transactions each month. The parameter \( \tau \) reflects the time that has elapsed since the last observed yield of the off-the-run issue.

Like the on-the-run issue, we linearize the measurement equation around the one-period forecast of \( l_t \) (i.e., \( l_{t-1} \)), and evaluate \( \Omega_t \left( l_{t-\tau}, \tau \right) \) at the contemporaneous prediction of \( l_{t-1} \):

\[
P_{\text{off}} = Z_0 + Z_h (l_t - l_{t-1}) + \varepsilon_{t,t} + E_{t-1}(\varepsilon_{t,t}, \varepsilon'_{t,t}) = U_{l,t}
\]  \tag{47}

where \( Z_0 \) and \( Z_h \) are the coefficients of the equation after linearization.

### 2.3 Defaultable straight bonds

A similar estimation procedure is applied to the corporate bond pricing model in (34a). The measurement and transition equations are

\[
P_d = g_d (h^*_t, \hat{l}_t, \hat{S}_t) + \varepsilon_{d,t}, \quad E_{t-1}(\varepsilon_{d,t}, \varepsilon'_{d,t}) = U_{d,t}
\]  \tag{48}

\[
h^*_t = d_d + \phi_t h^*_{t-1} + \eta_{d,t}, \quad E_{t-1}(\eta_{d,t}, \eta'_{d,t}) = Q_{d,t}
\]  \tag{49}

where

\[
d_d = \left[ \theta_d \left(1 - e^{-\kappa_d/12}\right) \right], \quad \phi_d = e^{-\kappa_d/12}
\]  \tag{50}
The function \( g_d(h_t^*, \hat{\tau}, \hat{S}_t) \) maps the default and liquidity intensities into corporate bond prices where \( \hat{S}_t \) is the vector of the predicted values of state variables. This mapping is implicitly given by numerically solving for the risky straight bond price implied by \( h_t^*, \hat{\tau}, \hat{S}_t \). The parameter \( \tau \) captures the time elapsed since the last observed yield of the corporate bond.

Similarly, the measurement equation is linearized around the one-period forecast of \( h_t^* \) (i.e., \( h_{t-1}^* \)), and \( \Omega_t(h_t^*, \tau) \) is evaluated at the contemporaneous prediction of \( h_{t-1}^* \):

\[
P_d = Z_{d0} + Z_{d1}(h_t^* - h_{t-1}^*) + \varepsilon_{d,t}, \quad E_{t-1}(\varepsilon_{d,t}, \varepsilon_{d,t}') = U_{d,t}
\]

where \( Z_{d0} \) and \( Z_{d1} \) are the coefficients of the equation after linearization.

### 3. Data and Empirical Estimation

#### 3.1 Data description

Data are obtained from GovPX, which consolidates quotes and transaction records from five of the six major inter-dealer Treasury securities brokers. About two-thirds of Treasury transactions in the secondary market were consummated among dealers through the interdealer market. Treasury dealers trade with one another mainly through interdealer brokers.

---

9 GovPX, Inc. was set up under the guidance of the Public Securities Association as a joint venture by the primary dealers and several interdealer brokers in 1991 to increase public access to U.S. Treasury security prices. GovPX consolidates and posts real-time quotes and trades data from five of the six major interdealer brokers (Liberty, Tullett & Tokyo, Garban, ICAP, and Hilliard and Farber), which together account for about two thirds of the interdealer broker market.

10 See Federal Reserve Bulletin (September 1993).
The GovPX dataset contains security identifier information, including the CUSIP, maturity date, as well as an indicator of whether the security is trading when-issued, on-the-run, or off-the-run. The quote data consist of the best bid and ask prices, associated yields, and respective bid and ask depths. The transaction data contain the time, initiator (buy or sell), price and trade size. Our sample period is from January 1993 to December 2002. September and October of 2001 are excluded due to the unstable market conditions after September 11.

Table I summarizes bond yields and trading characteristics such as bid-ask spreads, order imbalance, and depth calculated from all Treasury issues in the GovPX database over the sample period. These data are grouped by the seasonedness and the remaining time to maturity: 0-6 and 6-12 months, and 1-2, 2-5 and 5-10 years. In addition, all variables are arranged by monthly intervals in order to match the time interval of monthly corporate bond data series later on. Yields to maturity are obtained at monthly intervals for each Treasury security and averaged over the sample period. The bid-ask spread is the monthly average of the daily close ask minus bid divided by the midquote. Similarly, for each trade-related variable, we calculate the value for each day based on the intraday data and then take the average for each month. Order imbalance is the cumulative net order flows (purchase minus sell dollar volume) divided by the total dollar volume each day. Depth is the average of bid and ask sizes each day. Mean and standard deviations of these variables (monthly series) over the sample period are reported in Table I. As shown, net order flow is generally positive, indicating a net purchase. Yields and bid-ask spreads increase with maturity. The more liquid on-the-run
issues have lower spreads and higher depths. The spreads and trade-related variables will be used later to explain the liquidity spread of off-the-run issues.

We select only a subset of the Treasury database to estimate the pricing models of on- and off-the-run issues. Prices, coupon payments and time to maturity are inputs to the empirical model. We collect prices of each security at monthly intervals by selecting the last trading price in each month. For on-the-run issues, we employ data for the issues with maturities of 3 months, 6 months, 1, 2, 3, 5, and 10 years in empirical estimation. When a particular on-the-run issue becomes off the run, we replace it with another new on-the-run issue.

For off-the-run issues, we divide them into four categories by the length of period that the issue has been off the run: 0-3 months, 3-12 months, 1-2 years, 2 years and more. For each category and in each month, data are arranged by columns sorted by the remaining time to maturity of the Treasury in the range of \((m \text{ years}, (m+1) \text{ years}], m = 0, 1, 2,\ldots, 9\), respectively. By partitioning data in this manner, we treat off-the-run Treasuries with similar maturity as if they are homogeneous. Each month, we select only one Treasury issue in each category for given maturity that has a trade closest to the end of that particular month. For example, if there are three issues in each category and maturity with transaction records in month \(t\), we would select the one with a trade at the latest transaction time.\(^{11}\) This leaves only one bond issue in each maturity column in a particular month for each category. Off-the-run issues are less liquid and may not be traded each month. A data column is dropped if there are more than 20 missing monthly observations. This data selection procedure is applied to each off-the-run category (0-3

\(^{11}\) If there are more than one issue traded at the same time, we pick the one with the coupon rate closest to the on-the-run counterpart. If the coupon rates are all very close, we choose the one with median yield.
months, ..., and > 2 years) as well as the pooling sample combining all off-the-run issues. In empirical estimation, the actual maturity of the selected bond issue in each maturity column is used in model fitting.

Table II reports the summary statistics for price, coupon rate and maturity of on-the-run and off-the-run Treasury issues selected in the sample for estimating the pricing models. The average price of on-the-run issues is slightly lower than par while that of off-the-run issues is somewhat higher than par. Average coupon rates are from 3.4 to 7.2 percent while average maturity runs from 3.9 to 5.9 years.

Corporate bond data are obtained from two sources. The data from January 1993 to December 1996 are collected from the Lehman Brothers Fixed Income Database distributed by Warga (1998). This database includes monthly price (or yield), coupons, principal, accrued interest, ratings, maturity and callability on corporate bonds. To extend the sample period, we collect additional data from Lehman Brothers from January 1997 to December 2002. This dataset includes monthly index series of (par bond) yields to maturity, and maturity (by years) for each rating class from Aaa to B.\textsuperscript{12}

Individual corporate bonds with callable, puttable, convertible, subordinated and sinking-fund provisions and other embedded options are eliminated. Aaa bonds are excluded because of insufficient dealer-quoted data for this rating class. Bonds with an odd frequency of coupon payments are also excluded.\textsuperscript{13} Since there are not many bonds

\begin{footnotesize}
\begin{enumerate}
\item[\textsuperscript{12}] Although Lehman Brothers provides yields to maturity from 1 to 30 years, we use only the first ten years because there are very few corporate bond data from Warga’s tape with maturity more than 10 years.
\item[\textsuperscript{13}] We eliminate bonds that have unusually high pricing errors when priced using the estimated spot yield curve. We use $5 (per $100 face value) as the filter. Also, we eliminate bonds where the price or return data are obviously problematic. Bonds with extreme prices are an indication of recording errors and hence are removed. Moreover, we exclude bonds with less than 6-month maturity.
\end{enumerate}
\end{footnotesize}
with maturity longer than ten years in the Warga database, we focus on bonds with maturity less than ten years.

We use two data samples to estimate the pricing model of corporate bonds. The first sample is based on the individual bond data from the Warga database. We select only those firms that have at least 36 months in which at least two trader-quoted bond prices are observed. If firms have multiple bonds with similar maturities but different issuance dates, we select the most recently issued bond. If multiple bonds have the same most recently issued date, we select the one with the shortest maturity. The final sample for individual bond estimation includes 54 firms over the period January 1993 to December 1996. Table III reports the summary statistics of the individual corporate bond sample. Average coupon rates range from 5.96 to 9.98 percent. Average maturity in each rating class ranges from 5.2 to 6.5 years. Yields to maturity is from 6.43 to 10.70 percent.

The second sample combines the individual bond data with the Lehman Brothers’ index data. To combine with the Lehman Brothers’ index data, we obtain average bond price (yield), coupon rate, and maturity by ratings (from Aa to B) and maturity from the individual bond sample. We set maturity ranges to \((m \text{ years}, (m+1) \text{ years})\), \(m = 0, 1, 2, \ldots, 9\). Individual bonds are grouped by the maturity category and rating, and average price, coupon and maturity are calculated for each group in each month from January 1993 to December 1996. These average monthly series are then combined with Lehman Brothers’ index data to extend the sample period to December 2002. In empirical estimation, the actual average maturity of the bonds in each maturity-rating group is used to estimate the model.
3.2 Estimation for on-the-run Treasuries

Table IV reports the estimates for on-the-run Treasury issues. The extended Kalman filter method is employed to obtain the estimates of parameters and the two state vectors. As shown in Figure 1, the model fits the data very well. The root-mean-square-error (RMSE) is only 31 cents per $100 face value. All parameter estimates except for the volatility parameter of the first state factor $\sigma_i$ are significant at the five percent level.\(^{14}\)

The estimates of mean reversion parameters $\kappa_j$, $j = 1$ and $2$, are 0.6845 and 0.0317 respectively, and both are significant at the one percent level. However, the second factor is close to a martingale. Also, $2\kappa_j\theta_j > \sigma_j^2$ for both factors, suggesting that upward shifts are sufficiently large so that state variables do not revert to their initial values.\(^{15}\)

As expected, estimates of parameter $\eta_{si}$ are negative, indicating a positive risk premium required by investors. The mean reversion parameters under the equivalent martingale measure, $\kappa_{si} + \eta_{si}$, are positive, which implies that interest rate factors are stationary. For a single-factor CIR model, the forward rate volatility curve is upward sloping at the origin if $\kappa_{si} + \eta_{si} < 0$, whereas a positive value of $\kappa_{si} + \eta_{si}$ implies a hump-shaped forward rate volatility curve (see Jamshidian, 1995).

3.3 Estimation of off-the-run issues

The off-the-run issue is traded less frequently and thus has lower liquidity. Since investors are expected to require a compensation for this disadvantage, an off-the-run issue should trade at a higher yield than an otherwise identical on-the-run issue. We

\(^{14}\) Standard errors are calculated from QML (see Hamilton, 1994, p. 389).
\(^{15}\) Duffee (1999) reports similar results for a different sample period.
capture this liquidity difference by an intensity process. The estimates of the state variables from on-the-run issues $s_{it}, i = 1, 2$, are input into the pricing model of off-the-run issues via (6). Table V reports the parameter estimates of off-the-run Treasuries and the fitted values of the liquidity intensity process $\hat{I}_t$.

Similar to the estimation of the on-the-run pricing model, the Kalman filter fits the off-the-run data extremely well. The RMSEs are 0.42, 0.38, 0.50 and 0.49 per 100 face value for Treasuries off the run for 0-3 months, 3-12 months, 1-2 years and more than 2 years, respectively, and is only 0.44 for the combined off-the-run sample. The estimates of constant term $\alpha_i$ range from 4.45 to 6.75 bps. This is the component of liquidity spread that does not vary over time. As expected, $\alpha_i$ increases as the issue is more off the run, resulting in a higher level of liquidity spread. The estimates of $\kappa_i$ are 0.1124, 0.1070, 0.0864 and 0.1122 for the four off-the-run groups, respectively. The liquidity intensity processes are stationary for all off-the-run issues. The estimates of $\eta_i$ are all negative, suggesting that investors are compensated by a higher liquidity premium when a bond becomes less liquid. The absolute values of $\eta_i$ are all smaller than $\kappa_i$. The mean intensity $\theta_i$ tends to be higher for less liquid Treasury issues. The liquidity spread is 11.34 basis points for the pooling sample. The averages of estimated liquidity intensity $\hat{I}_t$ are quite close to $\theta_i$, and increase as the issue becomes more off the run (from 9.35 to 15.81 basis points). These estimates, together with $\alpha_i$ values, confirm that Treasury issues which are more off the run carry a higher liquidity risk premium. Figure 2 plots the liquidity intensities estimated by the model for four off-the-run groups. As shown,
liquidity intensity varies over time and both the level and volatility of the intensity has increased quite substantially in recent years.

The coefficients of interest rate factors, $\beta_1$ and $\beta_2$, capture the sensitivity of the liquidity intensity to interest rate factors. Estimates of both interest rate sensitivity parameters are negative indicating that liquidity spreads are inversely correlated with interest rate factors. However, the liquidity intensity is not very sensitive to the first interest rate factor. Taking the Treasury issue off the run for 0-3 months as an example, a 100 basis points (bps) increase in $s_1$ would only cause a decline of 0.34 bps in the liquidity spread.

Table VI reports the term structure of estimated liquidity spreads for off-the-run issues. Results show that the liquidity spreads between the on- and off-the-run Treasuries are not trivial. Liquidity spreads range from 7.72 to 11.01 bps for the issues of 6-month maturity, and 13.84 to 18.94 bps for the issues of 10-year maturity. For each maturity group, the liquidity spread increases as the period that the issue has been off the run is longer. For example, average liquidity spreads for the off-the-run issues with a remaining maturity of two years increase from 10.89 to 14.23 bps when Treasuries become more off the run.

The term structure of liquidity spreads is upward sloping; that is, for each off-the-run category, the liquidity spread increases with maturity. Although the term structure of liquidity spread is upward sloping, the slope of the spread curve does not necessarily get steeper as bonds become less liquid (or more off the run). This is due to mean reversion of the liquidity intensity process. As shown in Table IV, $\kappa_i + \eta_i > 0$ for off-the-run
issues, which indicates that the liquidity intensity does not always move higher due to mean reversion.

3.4 Regression of liquidity spreads on traditional liquidity measures

The literature has suggested several proxies for liquidity such as bid-ask spread, trading volume and depth. We next examine whether our estimates of liquidity spreads are linked to these liquidity variables. We consider four variables: bid-ask spread (BAS), abnormal volume (ABV), order imbalance (OIM) and depth (DEP). Abnormal volume (ABV) is the difference between the volume in the past month and the monthly average volume over the entire sample period divided by the average volume. The variable of OIM is order imbalance divided by the dollar volume each day and averaged for each month. The remaining two variables are as defined in Table I. Daily order imbalance, spreads and depth are averaged over the entire sample period for each selected off-the-run issue.

We select off-the-run Treasuries within one or two auctions from the GovPX database. These off-the-run issues are first divided into four categories according to the remaining time to maturity: 0-1 year, 1-2 years, 2-5 years and 5-10 years. Data for the four liquidity measures (BAS, ABV, OIM and DEP) are calculated for each off-the-run issue in a particular maturity category. In addition, the liquidity spreads are calculated for each issue based on the parameter estimates of off-the-run issues in Table V. We then regress liquidity spreads of the off-the-run issues in a maturity group against the four liquidity variables.

Table VII reports the results of cross-sectional regressions for each maturity group. Results show that liquidity spreads are significantly related to these liquidity
variables in most cases. The coefficients of bid-ask spread and order imbalance are positive as expected, since higher bid-ask spread and order imbalance imply lower liquidity. By contrast, higher abnormal volume and depth imply higher liquidity and so they are negatively associated with the liquidity spread. These findings are very encouraging given that estimated liquidity spreads are highly correlated with traditional liquidity measures. Thus, the estimated liquidity intensity from the prices of off-the-run issue appears to contain rich information for bond market liquidity.

3.5 Estimation of default and liquidity premia of corporate bonds

We now turn to the estimation of the corporate bond pricing model with liquidity risk. Although term structure of corporate bonds has received considerable attention in recent research, little is known about how liquidity risk affects corporate bond prices. Investors of corporate bonds worry about losing the principal once the firm defaults. With respect to liquidity risk, investors’ main concern is the possibility of being forced to sell the bond at a price substantially below its fair value, or pay high transaction costs to trade the security when liquidity is low. There is little doubt that investors will require a higher yield for bonds with higher liquidity risk. The question remaining unanswered is how much of the corporate bond spread can be attributed to liquidity risk.

In this section, we provide empirical estimates of the size of liquidity spread for corporate bonds with different ratings. We employ the estimated liquidity intensity for the combined off-the-run sample as a state factor in the term structure model of corporate bonds. We then jointly estimate the parameters of the liquidity intensity process with other term structure parameters from corporate bond price data.
Table VIII summarizes the estimates for corporate bonds. As expected, lower quality bonds have higher default probability. Mean default spreads ($\bar{h}'$) are 18.04, 34.21, 63.56, 98.44 and 177.04 basis points for Aa, A, Baa, Ba and B bonds, respectively. The implied default intensity exhibits a jump when the rating drops from the investment grade to the speculative grade. Default intensities are stationary as $\kappa_d + \eta_d > 0$ and the negative $\eta_d$ indicates a positive default risk premium.

The liquidity intensity coefficient $\omega$ is greater than one for bonds in all rating classes, confirming that corporate bonds are less liquid than Treasury bonds. In addition, $\omega$ increases as the rating decreases and more so for junk bonds. The estimated $\omega$ increases mildly from 1.18 to 2.09 when the rating drops from Aa to Baa but jumps to 4.32 for Ba bonds and 8.30 for B bonds. Results indicate that liquidity risk exposure is much higher for speculative-grade bonds. Overall, the model explains the behavior of corporate bond yields very well. As shown, the RMSE values are only 1.34 to 2.34 percent.

Table IX reports the components of corporate bond spreads. We first estimate the liquidity spread and then subtract it from the total estimated spread to obtain the nonliquidity component of spread. Results show that the liquidity spread is sizable and increases as the rating decreases. Thus, as a firm’s credit rating drops, its debt also becomes less liquid. The magnitude of the average liquidity spread ranges from 15 bps for Aa bonds to 182 for B bonds. These liquidity spread estimates are in line with previous studies. Longstaff et al. (2004) report 20 to 100 bps for bonds with ratings from AAA to BB. de Jong and Driessen (2004) also reports a similar range from 16 to 100 bps.
for AAA to BB bonds. Our estimates are 15 to 94 bps for Aa to Ba bonds, which are quite close to both studies. Another feature of empirical estimates is that liquidity spread is higher for longer-maturity bonds. For example, our liquidity spread estimates are 47, 110 and 209 bps for Baa, Ba and B bonds with ten-year maturity, respectively. de Jong and Driessen (2004) report a liquidity spread of 45 basis points for long-maturity investment-grade bonds. Again, our estimate for ten-year maturity bonds is quite close to theirs. Although the nonliquidity component is much larger, liquidity spread contributes significantly to the total spread. As shown, liquidity spreads explain 24 to 38 percent of the total spread for Aa to B bonds. Thus, including the liquidity risk factor into the term structure model of corporate bonds helps explain the yield spread puzzle.

The nonliquidity component of spread may include the effects of other factors such as taxes as indicated by Duffee (1999), although some studies have suggested that default premium accounts for most of the nonliquidity spread. The nonliquidity component increases from 48.3 bps to 293.7 bps as the rating of bonds decreases from Aa to B. Figures 3a and 3b plot the term structure of the nonliquidity and liquidity components of spreads for each rating class. As indicated, liquidity spreads increase with maturity. This positive relationship is stronger for lower-grade bonds.

Liquidity spread increases as credit risk increases and exhibits a jump when the bond rating falls below Baa to the junk bond category. For example, the mean liquidity spreads of Ba and B bonds are 94.3 and 182.4 bps compared to 38.1 bps for Baa bonds.

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16 An extreme case is when a firm defaults, all trading and lending activities will cease and there is no liquidity at all.
17 For example, Longstaff et al. (2004) suggests that the tax premium is minor compared to the default premium.
18 de Jong and Driessen (2004) report a similar finding that liquidity spread is higher for long-maturity bonds.
The proportion of liquidity spread to total spread averages about 25 percent for investment-grade bonds (Aa, A and Baa) but it increases to 33 percent for Ba bonds and 38 percent for B bonds. The liquidity spread is higher for junk bonds for two possible reasons. First, relative to investment-grade bonds, the size of junk bond issues is small. Second, many institutional investors such as pension and trust funds are prohibited from investing in speculative-grade bonds. This institutional constraint lowers the marketability, thereby further eroding the liquidity of junk bonds.

The results above are obtained based on the sample combining the individual bond data from the Warga tape and Lehman Brothers’ index data. We next estimate the term structure of defaultable bonds using only individual corporate bond data. Using individual corporate bond data in estimation avoids the potential problem associated with changes in the composition of bonds over time in the combined data sample. Since the Warga sample covers a shorter period, this estimation also provides a robustness check on the stability of parameter estimates over different sample periods.

Table X reports the mean, median and interquartile range of parameter estimates for individual corporate bonds. The range of parameter estimates is wider than that reported in Table VIII for the expanded sample. This is not surprising given that individual bond data reflect wider cross-firm variations. Default and liquidity parameter estimates in Table VIII fall within the range of individual bond estimates. Results show that parameter estimates using the expanded sample including Lehman Brothers’ index data are quite robust and the values of these parameters are stable over different sample periods.
Figure 4 plots the estimates of liquidity parameter $\omega$ and default intensities for individual bonds. As shown, liquidity and default intensities are positively correlated. Thus, lower quality bonds are accompanied by higher liquidity risk exposures.

The median estimate of liquidity spread in Table X falls between the estimates of Baa and Ba bonds in Table VIII. Average fitted default intensity is 0.64 percent, which is in line with previous estimates. Fons et al. (1994) report that the historical probability of a Baa-rated firm defaulting within 5 years is about 2%. Our estimate of default intensity is naturally higher because it is risk neutral. The estimates of the liquidity parameter $\omega$ range between 1.002 and 10.243 with a median of 2.779. Results show that on average liquidity risk of corporate bonds is much higher than off-the-run Treasury securities. Consistent with the findings in Table VIII, the nonliquidity component is much larger than the liquidity component. The mean liquidity premium accounts for about 29 percent of the total spread. The estimates of liquidity premium range from 10 percent to 49 percent of corporate bond spreads. The results confirm that liquidity premium is sizable for corporate bonds.

4. Conclusion

In this paper we adopt a reduced-form approach to model the liquidity of off-the-run Treasury securities. The proposed model characterizes the liquidity risk of off-the-run Treasuries and corporate bonds by an intensity process. As such, the pricing model is a straightforward extension of the pricing model of Duffie and Singleton (1999) and the closed-form pricing formula can be easily derived. Using this model, we examine how

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19 See, for example, Duffee (1999).
20 Since the liquidity intensity of Treasury bonds is negatively related to interest rate factors, the results suggest that the liquidity of corporate bonds is also negatively related to interest rate factors. To the extent
the market prices the liquidity risk of off-the-run Treasury issues relative to on-the-run issues. The model fits the individual Treasury bond data very well. Because of this, we are able to track the liquidity of off-the-run issues quite precisely. We then incorporate the estimated liquidity factor estimated from off-the-run Treasuries into the term structure model of defaultable bonds to estimate the liquidity components of corporate spreads.

We find that liquidity varies over time and the variation appears to be much higher in recent years. For Treasuries, liquidity spread increases as maturity increases and when the issues become more off the run. Estimated liquidity spreads of off-the-run issues are highly correlated with the traditional liquidity measure such as bid-ask spread, volume, order imbalance and depth. The results indicate that the model does a good job in extracting the liquidity information content from bond yields.

Liquidity risk is higher for corporate bonds and more so for speculative bonds. Liquidity spread of corporate bonds increases with maturity and credit risk. The magnitude of the liquidity premium is sizable. On average, liquidity spread is close to 30 percent of corporate bond spreads. Liquidity spread accounts for 25 percent of the spread for investment-grade bonds and 35 percent of that for speculative-grade bonds. Results show that including liquidity risk in the term structure model helps explain the corporate bond spread puzzle.

---

that interest rates are positively related to market performance, the results suggest that liquidity risk is low when the market is good which is consistent with the finding of Pastor and Stambaugh (2003).
Appendix

Extended Kalman Filter

From (36), the coupon bond price of an on-the-run issue is

\[ P_{on}(t, T) = f(S_t) = \sum_{i=0}^{T} CV_{on}(i, T, 0; S_t) + V_{on}(T, T, 0; S_t) \]  \tag{A1} 

In order to linearize it, we employ the Taylor expansion around the one-month-ahead forecast of the state vector \( S_t \), \( S_{t-1} = (s_{1,t-1}, s_{2,t-1}) \):

\[ Z_t = (Z_1, Z_2) = \left( \frac{\partial f}{\partial s_{1,t}}, \frac{\partial f}{\partial s_{2,t}} \right) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} \]  \tag{A2} 

\[ P_{on}(t, T) = Z_0 + Z_t(S_t - S_{t-1}) + \varepsilon_t, \quad E_{t-1}(\varepsilon_t \varepsilon_t') = U_t \]  \tag{A3} 

where \( f \) is the pricing function of on-the-run coupon bonds, and

\[ Z_1 = -C \sum_{i=0}^{T} \psi_1(i) V_{on}(i) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} - \psi_1(T) V_{on}(T) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} \]  \tag{A4} 

\[ Z_2 = -C \sum_{i=0}^{T} \psi_2(i) V_{on}(i) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} - \psi_2(T) V_{on}(T) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} \]  \tag{A5} 

\[ Z_0 = \sum_{i=0}^{T} CV_{on}(i, T, 0) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} + V_{on}(T, T, 0) \bigg|_{(s_{1,t}, s_{2,t})=(s_{1,t-1}, s_{2,t-1})} \]  \tag{A6} 

with the transition equation

\[ S_t = d + \phi S_{t-1} + \eta_t, \quad E_{t-1}(\eta_t \eta_t') = Q_t \]  \tag{A7} 

Prediction of the covariance matrix of state variables is given by,

\[ P_{t|t-1} = \phi P_{t-1} \phi' + Q_t \]  \tag{A8} 

Kalman’s gain matrix is

\[ P_t = P_{t|t-1} - P_{t|t-1} Z_t H_{t-1}^\top Z_t P_{t|t-1} \]  \tag{A9} 

The update equation of the state variable estimation is
\[ \hat{S}_t = d + \phi S_{t-1} + P_{t-1} Z_t - H^{-1}_{t-1} v_t \]  
(A10)

where \( v_t = P_{on} (t, T) - Z_0 \). Prediction of the state variables is given by

\[ S_{t-1} = d + \phi \hat{S}_{t-1} \]  
(A11)

The update equation of the covariance matrix of the state variables is

\[ H_t = Z_t P_{t-1} Z_t' + U_t \]  
(A12)

The log likelihood function is

\[ \log L = -\frac{NM}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{NM} \log |H_t| - \frac{1}{2} \sum_{t=1}^{NM} v_t' H_t^{-1} v_t \]  
(A13)

The estimation procedure for the intensity process associated with liquidity risk is similar. The linearized measurement equation is given by

\[ P_{off} = Z_{t0} + Z_{tq} (l_t - l_{t-1}) + \varepsilon_{t-1}, \quad E_{t-1} (\varepsilon_{t-1}, \varepsilon_{t-1}') = U_{t\varepsilon} \]  
(A14)

where

\[ Z_{t\varepsilon} = \frac{\delta g}{\delta l} \bigg|_{l_t= l_{t-1}} = C \sum_{i=t}^{T} \psi_{0} (i | V_{off} (i)) \bigg|_{l_t= l_{t-1}} - \psi_{0} (T | V_{off} (T)) \bigg|_{l_t= l_{t-1}} \]  
(A15)

\[ Z_{t0} = \sum_{i=t}^{T} CV_{off} (i | T, 0) \bigg|_{l_t= l_{t-1}} + V_{off} (T, T, 0) \bigg|_{l_t= l_{t-1}} \]

Similar procedures are applied to corporate bond pricing

\[ P_d = g_d (h_{t-1}^*, \hat{S}_t^*, \hat{S}_t), \quad E_{t-1} (\varepsilon_{d,t}, \varepsilon_{d,t}') = U_{d,t} \]  
(A16)

The measurement equation is linearized around one-period forecast of \( h_t^* \) (i.e., \( h_{t-1}^* \)), and \( \Omega_{t} (h_{t-1}^*, \tau) \) is evaluated at the contemporaneous prediction of \( h_{t-1}^* \):

\[ P_d = Z_{d0} + Z_{dt} (h_t^* - h_{t-1}^*) + \varepsilon_{d,t}, \quad E_{t-1} (\varepsilon_{d,t}, \varepsilon_{d,t}') = U_{d,t} \]  
(A17)

where
\[ Z_{dt} = \frac{\partial g_d}{\partial h_i} \bigg|_{h_i^* = h_{i-1}} = C \sum_{i=t}^{T} \psi_d(i) V_d(i) \bigg|_{h_i^* = h_{i-1}} - \psi_d(T) V_d(T) \bigg|_{h_i^* = h_{i-1}} \]
References


Pearson, N.C., and T.-S. Sun, 1994, The valuation of default risk in corporate bonds and interest rate swaps, working paper, University of Texas at Austin.


Note: This figure plots the fitted and observed prices for the on-the-run ten-year Treasury.
Figure 2
Liquidity Risk of Off-the-Run Treasuries

Note: This figure plots the estimated liquidity intensities for off-the-run Treasuries using monthly data from January 1993 to December 2002. Four curves represent groups with different length of time that the securities have been off the run.
Note: This figure plots the non-liquidity spreads against the remaining maturities for corporate bonds in five rating classes: Aa, A, Baa, Ba, and B. Spreads are expressed in basis points and maturities are in years.
Note: This figure plots the liquidity spread against the remaining maturity for corporate bonds in five rating classes: Aa, A, Baa, Ba, and B. Spreads are in basis points and maturities are in years.
Figure 4
Liquidity and Default Risks of Corporate Bonds

Note: This figure plots the relationship between the liquidity multiplier $\omega$ and implied default intensity $100 \cdot \frac{1}{\mu}$ for 54 individual firms. The sample period is from January 1993 to December 1996.
Table I
Summary Statistics of Liquidity Measures of Treasury Bonds

This table reports means and standard deviations of yields, net order flows, bid-ask spreads in percentage, and quoted depth over the period January 1993 through December 2002 for the Treasuries in the GovPX dataset, excluding September and October 2001 due to market instability surrounding the September 11 event. Treasuries are grouped by the remaining time-to-maturity: 0-6 months, 6-12 months, 1-2 years, 2-5 years and 5-10 years, and by seasonedness (on- and off-the-run). Yield to maturity is obtained at monthly intervals. Net Order Flow is obtained by first calculating the daily cumulative purchase minus sale orders (in dollar amount) and then taking the monthly average of the daily order flow. Bid-ask spread is the daily close ask minus bid divided by the midquote and averaged monthly. Quoted Depth is the average of bid and ask sizes (in million dollars) for both one- and two-sided quotes for each day. We calculate the average daily depth for each month. Yield and trade-related variables are averaged over the sample period and all Treasury issues.

<table>
<thead>
<tr>
<th>Seasonedness</th>
<th>Remaining Time to Maturity</th>
<th>0-6 months</th>
<th>6-12 months</th>
<th>1-2 years</th>
<th>2-5 years</th>
<th>5-10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yield to Maturity (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>On-the-Run</td>
<td></td>
<td>4.442</td>
<td>4.841</td>
<td>5.121</td>
<td>5.559</td>
<td>5.892</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.750)</td>
<td>(1.062)</td>
<td>(0.872)</td>
<td>(1.148)</td>
<td>(0.889)</td>
</tr>
<tr>
<td>Off-the-Run</td>
<td></td>
<td>4.498</td>
<td>4.901</td>
<td>5.222</td>
<td>5.707</td>
<td>6.089</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.767)</td>
<td>(1.050)</td>
<td>(0.9858)</td>
<td>(1.175)</td>
<td>(1.012)</td>
</tr>
<tr>
<td></td>
<td>Net Order Flow ($Mil.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>On-the-Run</td>
<td></td>
<td>41.6961</td>
<td>-29.974</td>
<td>118.824</td>
<td>146.231</td>
<td>108.492</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(220.030)</td>
<td>(205.715)</td>
<td>(237.088)</td>
<td>(454.810)</td>
<td>(374.101)</td>
</tr>
<tr>
<td>Off-the-Run</td>
<td></td>
<td>28.718</td>
<td>57.349</td>
<td>77.020</td>
<td>43.793</td>
<td>4.615</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(68.750)</td>
<td>(66.333)</td>
<td>(76.624)</td>
<td>(85.321)</td>
<td>(27.571)</td>
</tr>
<tr>
<td></td>
<td>Bid-Ask Spread (% of Midpoint Price)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>On-the-Run</td>
<td></td>
<td>0.027</td>
<td>0.056</td>
<td>0.113</td>
<td>0.1549</td>
<td>0.3003</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.012)</td>
<td>(0.025)</td>
<td>(0.054)</td>
<td>(0.079)</td>
<td>(0.067)</td>
</tr>
<tr>
<td>Off-the-Run</td>
<td></td>
<td>0.057</td>
<td>0.135</td>
<td>0.224</td>
<td>0.368</td>
<td>0.514</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.066)</td>
<td>(0.089)</td>
<td>(0.102)</td>
<td>(0.124)</td>
<td>(0.153)</td>
</tr>
<tr>
<td></td>
<td>Quoted Depth ($Mil.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.843)</td>
<td>(3.254)</td>
<td>(4.279)</td>
<td>(6.710)</td>
<td>(6.085)</td>
</tr>
<tr>
<td>Off-the-Run</td>
<td></td>
<td>7.480</td>
<td>5.773</td>
<td>2.740</td>
<td>1.778</td>
<td>2.046</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.624)</td>
<td>(3.793)</td>
<td>(2.285)</td>
<td>(1.802)</td>
<td>(1.137)</td>
</tr>
</tbody>
</table>
Table II
Summary Statistics of On-the-Run and Off-the-Run Treasury Issues

Monthly data are obtained over the period from January 1993 to December 2002 excluding September and October 2001 due to market instability surrounding the September 11 event. Off-the-run data are divided into five categories according to the length of time that the bond has been off the run, namely, 0-3 months, 3-12 months, 1-2 years and two years and more. In each month during the sample period, prices, coupons and maturities of off-the-run issues are divided into columns sorted by the remaining time to maturity, \( (m \text{ years}, (m+1) \text{ years}), m = 0, 1, 2, \ldots, 9 \). A column is dropped if the total number of months with missing observations is more than 20.

<table>
<thead>
<tr>
<th>NOB</th>
<th>Price (per $100 principle)</th>
<th>Coupon (per $100 principle)</th>
<th>Maturity (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
<td>Mean</td>
<td>Max</td>
</tr>
<tr>
<td>On-the-run</td>
<td>826</td>
<td>90.06</td>
<td>99.62</td>
</tr>
<tr>
<td>Off-the-run (0-3 months)</td>
<td>540</td>
<td>93.24</td>
<td>99.25</td>
</tr>
<tr>
<td>Off-the-run (3-12 months)</td>
<td>540</td>
<td>92.08</td>
<td>100.14</td>
</tr>
<tr>
<td>Off-the-run (1-2 years)</td>
<td>464</td>
<td>87.64</td>
<td>101.46</td>
</tr>
<tr>
<td>Off-the-run (&gt;2 years)</td>
<td>432</td>
<td>92.29</td>
<td>103.94</td>
</tr>
<tr>
<td>Off-the-run (combined)</td>
<td>708</td>
<td>87.64</td>
<td>100.03</td>
</tr>
</tbody>
</table>
Table III
Summary Statistics of Individual Corporate Bonds

Monthly data are obtained over the period from January 1993 to December 1996. 54 firms are selected in the final sample. Bond data are categorized into five ratings: Aa, A, Baa, Ba, and B. The table reports mean prices, coupons, maturities, yields-to-maturity (YTM) and corporate bond spreads relative to the yields of on-the-run Treasury securities with the same maturity.

<table>
<thead>
<tr>
<th>Bond ratings</th>
<th>Numb. of Firms</th>
<th>Price (per $100 principle)</th>
<th>Coupon (per $100 principle)</th>
<th>Maturity (years)</th>
<th>YTM (percent)</th>
<th>Spread (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aa</td>
<td>6</td>
<td>100.26</td>
<td>5.96</td>
<td>5.94</td>
<td>6.43</td>
<td>67.64</td>
</tr>
<tr>
<td>A</td>
<td>13</td>
<td>99.64</td>
<td>6.69</td>
<td>5.22</td>
<td>6.96</td>
<td>119.97</td>
</tr>
<tr>
<td>Baa</td>
<td>17</td>
<td>101.04</td>
<td>6.57</td>
<td>5.29</td>
<td>7.31</td>
<td>155.44</td>
</tr>
<tr>
<td>Ba</td>
<td>12</td>
<td>102.14</td>
<td>8.11</td>
<td>5.77</td>
<td>8.44</td>
<td>268.64</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>98.28</td>
<td>9.98</td>
<td>6.54</td>
<td>10.70</td>
<td>494.50</td>
</tr>
</tbody>
</table>
Table IV
Estimation of On-the-Run Issues Using the Extended Kalman Filter

This table reports the estimate of the on-the-run model. The short rate process is governed by

\[ r_t = \alpha + s_{1,t} + s_{2,t} \]

where the two factors follow the dynamics

\[ ds_{i,t} = \kappa_i (\theta_i - s_{i,t}) dt + \sigma_i \sqrt{s_{i,t}} dZ_{i,t}^i \quad \text{for} \quad i = 1, 2 \quad \text{(physical measure)} \]

\[ ds_{i,t} = (\kappa_{s_i} \theta_{s_i} - (\kappa_{s_i} + \eta_{s_i}) s_{i,t}) dt + \sigma_{s_i} \sqrt{s_{i,t}} d\tilde{Z}_{i,t}^s \quad \text{for} \quad i = 1, 2 \quad \text{(risk-neutral measure)} \]

Following Duffee (1999), the value of \( \alpha \) is fixed at -1.00. Standard errors are in the parentheses. The sample period is from January 1993 to December 2002, excluding September and October 2001.

<table>
<thead>
<tr>
<th></th>
<th>( \kappa_j )</th>
<th>( \theta_j )</th>
<th>( \eta_{sj} )</th>
<th>( \sigma_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6845</td>
<td>1.1818</td>
<td>-0.0134</td>
<td>0.0103</td>
</tr>
<tr>
<td></td>
<td>(0.0549)</td>
<td>(0.0245)</td>
<td>(0.0010)</td>
<td>(0.0126)</td>
</tr>
<tr>
<td>2</td>
<td>0.0317</td>
<td>0.0711</td>
<td>-0.0129</td>
<td>0.0464</td>
</tr>
<tr>
<td></td>
<td>(0.0051)</td>
<td>(0.0216)</td>
<td>(0.0010)</td>
<td>(0.0122)</td>
</tr>
</tbody>
</table>

RMSE (per $100 face value) 0.31
Table V

Estimation of Off-the-Run Issues Using the Extended Kalman Filter

This table reports the estimate of the off-the-run term structure model with liquidity risk. The intensity process of liquidity is given by

\[ \zeta_t = \alpha_t + l_t + \beta_1 (s_{1,t} - \bar{s}_{1,t}) + \beta_2 (s_{2,t} - \bar{s}_{2,t}) \]

where \( l_t \) follows a single-factor square-root dynamics

\[
\begin{align*}
dl_i &= \kappa_i (\theta_i - l_i)dt + \sigma_i \sqrt{l_i} dZ_i \\
&= (\kappa_i \theta_i - (\kappa_i + \eta_i) l_i)dt + \sigma_i \sqrt{l_i} d\hat{Z}_i
\end{align*}
\]

(physical measure)

where \( \kappa_i, \theta_i, \eta_i, \sigma_i, \beta_1, \beta_2 \) are estimated parameters. Standard errors are in the parentheses. The sample period is from January 1993 to December 2002, excluding September and October 2001.

Panel A: Off-the-run issues: the whole sample

<table>
<thead>
<tr>
<th>(100^*\alpha_i)</th>
<th>(\kappa_i)</th>
<th>(\theta_i)</th>
<th>(\eta_i)</th>
<th>(\sigma_i)</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(100^*\bar{T})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0495</td>
<td>0.1598</td>
<td>0.0011</td>
<td>-0.0558</td>
<td>0.0076</td>
<td>-0.0080</td>
<td>-0.0307</td>
<td>0.1134</td>
</tr>
<tr>
<td>(0.0122)</td>
<td>(0.0090)</td>
<td>(0.0005)</td>
<td>(0.0022)</td>
<td>(0.0014)</td>
<td>(0.0007)</td>
<td>(0.0021)</td>
<td></td>
</tr>
</tbody>
</table>

RMSE (per $100 face value) 0.44

Panel B: Off-the-run issues: 0-3 months

<table>
<thead>
<tr>
<th>(100^*\alpha_i)</th>
<th>(\kappa_i)</th>
<th>(\theta_i)</th>
<th>(\eta_i)</th>
<th>(\sigma_i)</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(100^*\bar{T})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0445</td>
<td>0.1123</td>
<td>0.0010</td>
<td>-0.0865</td>
<td>0.0059</td>
<td>-0.0034</td>
<td>-0.0277</td>
<td>0.0935</td>
</tr>
<tr>
<td>(0.0084)</td>
<td>(0.0073)</td>
<td>(0.0003)</td>
<td>(0.0043)</td>
<td>(0.0022)</td>
<td>(0.0010)</td>
<td>(0.0017)</td>
<td></td>
</tr>
</tbody>
</table>

RMSE (per $100 face value) 0.42

Panel C: Off-the-run issues: 3-12 months

<table>
<thead>
<tr>
<th>(100^*\alpha_i)</th>
<th>(\kappa_i)</th>
<th>(\theta_i)</th>
<th>(\eta_i)</th>
<th>(\sigma_i)</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(100^*\bar{T})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0463</td>
<td>0.1070</td>
<td>0.0015</td>
<td>-0.0139</td>
<td>0.0045</td>
<td>-0.0024</td>
<td>-0.0322</td>
<td>0.1277</td>
</tr>
<tr>
<td>(0.0150)</td>
<td>(0.0438)</td>
<td>(0.0004)</td>
<td>(0.0058)</td>
<td>(0.0039)</td>
<td>(0.0011)</td>
<td>(0.0053)</td>
<td></td>
</tr>
</tbody>
</table>

RMSE (per $100 face value) 0.38

Panel D: Off-the-run issues: 1-2 years

<table>
<thead>
<tr>
<th>(100^*\alpha_i)</th>
<th>(\kappa_i)</th>
<th>(\theta_i)</th>
<th>(\eta_i)</th>
<th>(\sigma_i)</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(100^*\bar{T})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0501</td>
<td>0.0864</td>
<td>0.0015</td>
<td>-0.0895</td>
<td>0.0059</td>
<td>-0.0032</td>
<td>-0.0269</td>
<td>0.1423</td>
</tr>
<tr>
<td>(0.0185)</td>
<td>(0.0255)</td>
<td>(0.0004)</td>
<td>(0.0401)</td>
<td>(0.0029)</td>
<td>(0.0010)</td>
<td>(0.0052)</td>
<td></td>
</tr>
</tbody>
</table>

RMSE (per $100 face value) 0.50
Panel E: Off-the-run issues: more than 2 years

<table>
<thead>
<tr>
<th>$100*\alpha_i$</th>
<th>$\kappa_i$</th>
<th>$\theta_i$</th>
<th>$\eta_i$</th>
<th>$\sigma_i$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$100*\bar{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0675</td>
<td>0.1123</td>
<td>0.0018</td>
<td>-0.0137</td>
<td>0.0075</td>
<td>-0.0029</td>
<td>-0.0158</td>
<td>0.1581</td>
</tr>
<tr>
<td>(0.0258)</td>
<td>(0.0202)</td>
<td>(0.0004)</td>
<td>(0.0028)</td>
<td>(0.0034)</td>
<td>(0.0030)</td>
<td>(0.0074)</td>
<td></td>
</tr>
</tbody>
</table>

RMSE (per $100$ face value) 0.49
Table VI
Average Implied Liquidity Spreads

This table reports the estimates of liquidity spreads. For an on-the-run zero-coupon bond, the price is
\[ V_{on}(t, T, 0) = \exp[\psi_0(t) - s_1, \psi_1(t) - s_2, \psi_2(t)] \]
and the spot yield is
\[ Y_{on}(t, T) = \frac{-\psi_0(t) + \hat{s}_1, \psi_1(t) + \hat{s}_2, \psi_2(t)}{T - t} \]
The price of an off-the-run zero-coupon bond is
\[ V_{off}(t, T, 0) = \exp[\psi_0(t) - s^*, \psi_1(t) - s^*, \psi_2(t) - I, \psi_1(t)] \]
and its spot yield is
\[ Y_{off}(t, T) = \frac{-\psi_0(t) + \hat{s}_1, \psi_1(t) + \hat{s}_2, \psi_2(t) + \hat{l}, \psi_1(t)}{T - t} \]
The liquidity spread (in basis points) is calculated as the off-the-run spot yield less the on-the-run spot yield.

<table>
<thead>
<tr>
<th>Off-the-run period</th>
<th>6 months</th>
<th>1 year</th>
<th>2 years</th>
<th>5 years</th>
<th>10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3 months</td>
<td>7.72</td>
<td>8.50</td>
<td>10.89</td>
<td>12.76</td>
<td>13.84</td>
</tr>
<tr>
<td>3-12 months</td>
<td>8.74</td>
<td>10.68</td>
<td>11.43</td>
<td>13.47</td>
<td>14.37</td>
</tr>
<tr>
<td>1-2 years</td>
<td>10.35</td>
<td>11.08</td>
<td>12.01</td>
<td>14.86</td>
<td>15.55</td>
</tr>
<tr>
<td>2 years &amp; more</td>
<td>11.01</td>
<td>12.64</td>
<td>14.23</td>
<td>17.08</td>
<td>18.94</td>
</tr>
</tbody>
</table>
Table VII
Regression of Implied Liquidity Spreads on Traditional Liquidity Measures
for Off-the-Run Treasuries

Panel A presents results of regression of implied liquidity spreads of off-the-run Treasury issues on four liquidity measures, BAS, ABV, OIM and DEP. Off-the-run Treasury data from the GovPX are first divided into four categories according to the remaining time to maturity, namely 0-1 year, 1-2 years, 2-5 years and 5-10 years. In each maturity category, data for four liquidity measures are obtained by tracking the issues that are off the run within one or two auctions. BAS and DEP are the bid-ask spread and quoted depth as defined in Table II. We calculate total trading volume for each day and then average over the month. Order imbalance (OIM) is obtained by first calculating the daily cumulative net order flows (purchase orders minus sell orders in dollars) divided by total daily dollar volume and then taking the monthly average of the absolute order imbalance. Abnormal volume (ABV) is the difference between the volume in the past month and the monthly average volume divided by the monthly average volume. BAS, ABV, OIM and DEP are calculated for each off-the-run issue. The implied liquidity spreads for each month are calculated for each off-the-run issue with given maturity in each group using the parameter estimates of the reduced-form model I Table V. The sample period is from January 1993 to December 2002. Cross-sectional regression results are reported by groups of maturities: 0-1 year, 1-2 years, 2-5 years and 5-10 years, while t-values are in the parentheses. Panel B reports the correlation matrix of the variables.

Panel A: Regressions of implied liquidity spreads on traditional liquidity measures

<table>
<thead>
<tr>
<th>Maturities</th>
<th>Intercept</th>
<th>BAS</th>
<th>ABV</th>
<th>OIM</th>
<th>DEP</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1 year</td>
<td>0.0298</td>
<td>1.2190</td>
<td>-0.0365</td>
<td>0.0822</td>
<td>-0.0030</td>
<td>0.7925</td>
</tr>
<tr>
<td></td>
<td>(1.87)</td>
<td>(4.56)</td>
<td>(2.38)</td>
<td>(2.03)</td>
<td>(3.66)</td>
<td></td>
</tr>
<tr>
<td>1-2 years</td>
<td>0.0138</td>
<td>0.8537</td>
<td>-0.0380</td>
<td>0.0877</td>
<td>-0.0058</td>
<td>0.7332</td>
</tr>
<tr>
<td></td>
<td>(2.47)</td>
<td>(4.70)</td>
<td>(2.28)</td>
<td>(1.98)</td>
<td>(3.49)</td>
<td></td>
</tr>
<tr>
<td>2-5 years</td>
<td>0.0132</td>
<td>0.5460</td>
<td>-0.0435</td>
<td>0.0813</td>
<td>-0.0043</td>
<td>0.7277</td>
</tr>
<tr>
<td></td>
<td>(4.30)</td>
<td>(4.25)</td>
<td>(1.09)</td>
<td>(2.11)</td>
<td>(3.02)</td>
<td></td>
</tr>
<tr>
<td>5-10 years</td>
<td>0.0292</td>
<td>0.4102</td>
<td>-0.0681</td>
<td>0.0866</td>
<td>-0.0062</td>
<td>0.6508</td>
</tr>
<tr>
<td></td>
<td>(2.86)</td>
<td>(7.23)</td>
<td>(1.43)</td>
<td>(1.64)</td>
<td>(1.97)</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: Correlation matrix

<table>
<thead>
<tr>
<th></th>
<th>BAS</th>
<th>ABV</th>
<th>OIM</th>
<th>DEP</th>
</tr>
</thead>
<tbody>
<tr>
<td>BAS</td>
<td>1.0000</td>
<td>0.1436</td>
<td>0.0045</td>
<td>-0.1060</td>
</tr>
<tr>
<td>ABV</td>
<td>0.1436</td>
<td>1.0000</td>
<td>0.0267</td>
<td>-0.0022</td>
</tr>
<tr>
<td>OIM</td>
<td>0.0045</td>
<td>0.0267</td>
<td>1.0000</td>
<td>0.0011</td>
</tr>
<tr>
<td>DEP</td>
<td>-0.1060</td>
<td>-0.0022</td>
<td>0.0011</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Table VIII
Estimation of the Corporate Bond Model

This table reports the parameter estimates of the defaultable bond model using the combined sample. Standard errors are in the parentheses.

Panel A: Aa Bonds (RMSE $1.53)

<table>
<thead>
<tr>
<th>$100 \cdot \alpha_d$</th>
<th>$\kappa_d$</th>
<th>$\theta_d$</th>
<th>$\eta_d$</th>
<th>$\sigma_d$</th>
<th>$\beta_{1,d}$</th>
<th>$\beta_{2,d}$</th>
<th>$\omega$</th>
<th>$100 \cdot \bar{h}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2753</td>
<td>0.4543</td>
<td>0.0017</td>
<td>-0.0946</td>
<td>0.1532</td>
<td>-0.0340</td>
<td>-0.0104</td>
<td>1.1761</td>
<td>0.1804</td>
</tr>
<tr>
<td>(0.7771)</td>
<td>(0.1553)</td>
<td>(0.0007)</td>
<td>(0.0253)</td>
<td>(0.0446)</td>
<td>(0.0123)</td>
<td>(0.0301)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Panel B: A Bonds (RMSE $1.34)

<table>
<thead>
<tr>
<th>$100 \cdot \alpha_d$</th>
<th>$\kappa_d$</th>
<th>$\theta_d$</th>
<th>$\eta_d$</th>
<th>$\sigma_d$</th>
<th>$\beta_{1,d}$</th>
<th>$\beta_{2,d}$</th>
<th>$\omega$</th>
<th>$100 \cdot \bar{h}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3119</td>
<td>0.4028</td>
<td>0.0030</td>
<td>-0.2125</td>
<td>0.1789</td>
<td>-0.0550</td>
<td>-0.0216</td>
<td>1.4231</td>
<td>0.3421</td>
</tr>
<tr>
<td>(0.1033)</td>
<td>(0.0724)</td>
<td>(0.0011)</td>
<td>(0.0591)</td>
<td>(0.0873)</td>
<td>(0.0131)</td>
<td>(0.0097)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Panel C: Baa Bonds (RMSE $2.15)

<table>
<thead>
<tr>
<th>$100 \cdot \alpha_d$</th>
<th>$\kappa_d$</th>
<th>$\theta_d$</th>
<th>$\eta_d$</th>
<th>$\sigma_d$</th>
<th>$\beta_{1,d}$</th>
<th>$\beta_{2,d}$</th>
<th>$\omega$</th>
<th>$100 \cdot \bar{h}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5181</td>
<td>0.4080</td>
<td>0.0064</td>
<td>-0.1528</td>
<td>0.1544</td>
<td>-0.1222</td>
<td>-0.0125</td>
<td>2.0893</td>
<td>0.6356</td>
</tr>
<tr>
<td>(0.1793)</td>
<td>(0.1513)</td>
<td>(0.0011)</td>
<td>(0.0506)</td>
<td>(0.0706)</td>
<td>(0.0558)</td>
<td>(0.0074)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$100 \cdot \alpha_d$</td>
<td>$\kappa_d$</td>
<td>$\theta_d$</td>
<td>$\eta_d$</td>
<td>$\sigma_d$</td>
<td>$\beta_{1,d}$</td>
<td>$\beta_{2,d}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>------------------</td>
<td>-----------------------</td>
<td>------------</td>
<td>------------</td>
<td>-----------</td>
<td>------------</td>
<td>----------------</td>
<td>----------------</td>
<td>---------</td>
</tr>
<tr>
<td><strong>Panel D: Ba Bonds (RMSE $2.27$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.9461</td>
<td>0.4331</td>
<td>0.0118</td>
<td>-0.1934</td>
<td>0.2138</td>
<td>-0.0729</td>
<td>-0.0318</td>
<td>4.3197</td>
</tr>
<tr>
<td></td>
<td>(0.3352)</td>
<td>(0.1255)</td>
<td>(0.0030)</td>
<td>(0.0456)</td>
<td>(0.0906)</td>
<td>(0.0425)</td>
<td>(0.0166)</td>
<td></td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Panel E: B Bonds (RMSE $2.34$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.2501</td>
<td>0.3705</td>
<td>0.0171</td>
<td>-0.2443</td>
<td>0.2654</td>
<td>-0.0895</td>
<td>-0.0395</td>
<td>8.3020</td>
</tr>
<tr>
<td></td>
<td>(0.3170)</td>
<td>(0.0717)</td>
<td>(0.0045)</td>
<td>(0.0732)</td>
<td>(0.0644)</td>
<td>(0.1124)</td>
<td>(0.0157)</td>
<td></td>
</tr>
</tbody>
</table>
Table IX
Implied Liquidity and Nonliquidity Components of Corporate Bond Spreads

Estimates of spreads are decomposed into liquidity and nonliquidity components. We calculate the liquidity spread first and define the remaining spread as the nonliquidity spread which may include the effects of other factors such as taxes. Spreads are estimated for bonds in each rating class (Aa, A, Baa, Ba and B) over ten maturities (1-10 years). This table reports the mean spread components for each rating class in both basis points and percentage of the estimated total spread.

<table>
<thead>
<tr>
<th>Bond Rating</th>
<th>Estimated Total Spread</th>
<th>Nonliquidity Component</th>
<th>Liquidity Component</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Premium (bps)</td>
<td>Percentage of Spread</td>
</tr>
<tr>
<td>Aa</td>
<td>63.4</td>
<td>48.3</td>
<td>76.2%</td>
</tr>
<tr>
<td>A</td>
<td>85.3</td>
<td>64.7</td>
<td>75.9%</td>
</tr>
<tr>
<td>Baa</td>
<td>150.3</td>
<td>114.1</td>
<td>75.9%</td>
</tr>
<tr>
<td>Ba</td>
<td>290.4</td>
<td>198.0</td>
<td>68.2%</td>
</tr>
<tr>
<td>B</td>
<td>480.0</td>
<td>293.7</td>
<td>61.2%</td>
</tr>
</tbody>
</table>
Table X
Estimates of Individual Bonds

Parameter estimates and spread components of corporate bonds are summarized in this table. The model is estimated using the individual bond data sample including 54 firms over the period January 1993 to December 1996. The liquidity and nonliquidity components are reported in percentage of the total spread.

\[
\begin{array}{cccccccccccc}
100 \cdot \alpha_d & \kappa_d & 100 \cdot \theta_d & \eta_d & \sigma_d & \beta_{1,d} & \beta_{2,d} & \omega & 100 \cdot \bar{h}^* & \text{Liquidity spread (bps)} & \text{Liquidity spread (percent)} \\
\hline
1^{st} \text{quartile} & 0.4103 & 0.3413 & 0.4681 & -0.3224 & 0.1669 & -0.0745 & -0.0377 & 1.5578 & 0.3778 & 21.58 & 23.89\% \\
median & 0.6260 & 0.5230 & 0.6888 & -0.2740 & 0.2051 & -0.0434 & -0.0176 & 2.7794 & 0.6444 & 63.47 & 25.31\% \\
3^{rd} \text{quartile} & 1.0261 & 0.6817 & 1.1029 & -0.1957 & 0.3283 & -0.0271 & -0.0114 & 4.2589 & 1.1435 & 92.11 & 34.10\% \\
mean & 0.7253 & 0.4862 & 0.7932 & -0.2761 & 0.2146 & -0.0428 & -0.0159 & 3.4806 & 0.7843 & 67.65 & 28.51\% \\
max & 1.9475 & 0.7867 & 1.9682 & -0.1171 & 0.4948 & -0.0104 & 0.0243 & 10.2433 & 1.5525 & 204.30 & 48.97\% \\
min & 0.1372 & 0.1973 & 0.1470 & -0.4539 & 0.0845 & -0.306 & -0.1104 & 1.0025 & 0.1305 & 9.04 & 9.64\% \\
\end{array}
\]