Mr. Madoff's Amazing Returns: An Analysis of the Split-Strike Conversion Strategy

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Abstract

It is now known that the very impressive investment returns generated by Bernie Madoff were based on a sophisticated Ponzi scheme. Madoff claimed to use a split-strike conversion strategy. This strategy consists of a long equity position plus a long put and a short call. In this paper we examine Madoff's returns and compare his investment performance with what could have been obtained using a split-strike conversion strategy based on the historical data. We also analyze the split-strike strategy in general and derive expressions for the expected return, standard deviation, Sharpe ratio and correlation with the market of this strategy. We find that Madoff's returns lie well outside their theoretical bounds and should have raised suspicions about Madoff's performance.

Keywords: Madoff, split-strike conversion strategy, performance measurement, derivatives.

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1 Introduction

In December 2008, the investment operation of Bernie Madoff was exposed as a giant Ponzi scheme[1]. Madoff had attracted a wide following because he delivered consistently high returns with very low volatility over a long period. He claimed to use a split-strike conversion strategy to obtain these low risk returns. This strategy involves taking a long position in equities together with a short call and a long put on an equity index to lower the volatility of the position. We know now that these returns were fictitious. The Madoff affair raises the obvious questions as to why it was not discovered earlier and why investors and regulators missed the various red flags. A number of these points are discussed in Markopoulos (2009) and Gregoriou and Lhabitant (2009). The paper of Clauss, Roncalli and Weisang (2009) complements our work. It focuses on risk management implications of Madoff's fraud, in particular on the lack of regulation and proposes to improve capital requirements for operational risk.

This paper discusses certain aspects of the split-strike strategy and analyzes the reported performance of Madoff's funds. We analyze the Fairfield Sentry Ltd hedge fund which was one of Madoff's feeder funds. Fairfield Sentry describes its strategy as follows.

> *The Fund seeks to obtain capital appreciation of its assets principally through the utilization of a nontraditional options strategy described as a split-strike conversion to which the Fund allocates the predominant portion of its assets. The investment strategy has defined risk and reward parameters. The establishment of a typical position entails (i) the purchase of a group or basket of securities that are intended to highly correlate to the S&P 100 Index, (ii) the purchase of out-of-the-money S&P 100 Index put options with a notional value approximately equal to the market value of the basket of equity securities and (iii) the sale of out-of-the-money S&P 100 Index call options with a notional value approximately equal to the market value of the basket of equity securities. The basket typically*

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[1] In a Ponzi Scheme, returns to investors came from their own money or money paid by subsequent investors rather than from any actual profit earned. See a discussion on Ponzi Schemes in Bhattacharya (2003).

> *consists of 40-50 stocks in the S&P 100 Index. The primary purpose of the long put options is to limit the market risk of the stock basket at the strike price of the long puts. The primary purpose of the short call options is to largely finance the cost of the put hedge and increase the stand-still rate of return. The "split-strike conversion" strategy*[2] *is implemented by Bernie L. Madoff Investment Securities LLC("BLM"), a broker dealer registered with the Securities and Exchange Commission through accounts maintained by the Fund in that firm. The services of BLM and its personnel are essential to the continued operation of the Fund and its profitability.*

In the next section we analyze the performance of Fairfield Sentry for the period December 1990 to October 2008. The most dramatic aspect of the performance is the very low volatility of the returns. This in turn leads to an unusually high Sharpe ratio. We compare these returns with what could have been obtained by following a split-strike conversion strategy in real time using the actual historical returns and find that while the expected return is plausible, the volatility of the strategy in practice is much higher. Section Three analyzes some theoretical properties of the distribution of returns of the split-strike strategy. In particular we develop closed-form expressions for the expected return, standard deviation, correlation and Sharpe ratio of this strategy. Section Four illustrates numerically the properties of the split-strike strategy.

2 Analysis of the Empirical Results

In this section we analyze the Fairfield Sentry return performance and contrast the reported returns with those that could be achieved using a split-strike conversion strategy. The reported monthly returns for the period December 1990 to October 2008 are given in Table 1. These returns are amazingly consistent with an exceptionally low volatility. The monthly volatility is 71 basis points corresponding to an annual volatility of 2.45%. The average monthly return for the strategy 84 basis points corresponding to an annual average return of 10.59%. Investors clearly put a very high value on this combination of high returns and low volatility.

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[2] This is a marketing name for a collar strategy. In other words, this is commonly called a bearish collar strategy.

Table 1: **Fairfield Sentry monthly returns from Dec. 1990 to Oct. 2008.**

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2008	0.63	0.06	0.18	0.93	0.81	-0.06	0.72	0.71	0.50	-0.06	na	na
2007	0.29	-0.11	1.64	0.98	0.81	0.34	0.17	0.31	0.97	0.46	1.04	0.23
2006	0.70	0.20	1.31	0.94	0.70	0.51	1.06	0.77	0.68	0.42	0.86	0.86
2005	0.51	0.37	0.85	0.14	0.63	0.46	0.13	0.16	0.89	1.61	0.75	0.54
2004	0.88	0.44	-0.01	0.37	0.59	1.21	0.02	1.26	0.46	0.03	0.79	0.24
2003	-0.35	-0.05	1.85	0.03	0.9	0.93	1.37	0.16	0.86	1.26	-0.14	0.25
2002	-0.04	0.53	0.39	1.09	2.05	0.19	3.29	-0.13	0.06	0.66	0.09	0.00
2001	2.14	0.08	1.07	1.26	0.26	0.17	0.38	0.94	0.66	1.22	1.14	0.12
2000	2.14	0.13	1.77	0.27	1.30	0.73	0.58	1.26	0.18	0.86	0.62	0.36
1999	1.99	0.11	2.22	0.29	1.45	1.70	0.36	0.87	0.66	1.05	1.54	0.32
1998	0.85	1.23	1.68	0.36	1.69	1.22	0.76	0.21	0.98	1.86	0.78	0.26
1997	2.38	0.67	0.80	1.10	0.57	1.28	0.68	0.28	2.32	0.49	1.49	0.36
1996	1.42	0.66	1.16	0.57	1.34	0.15	1.86	0.20	1.16	1.03	1.51	0.41
1995	0.85	0.69	0.78	1.62	1.65	0.43	1.02	-0.24	1.63	1.53	0.44	1.03
1994	2.11	-0.44	1.45	1.75	0.44	0.23	1.71	0.35	0.75	1.81	-0.64	0.60
1993	-0.09	1.86	1.79	-0.01	1.65	0.79	0.02	1.71	0.28	1.71	0.19	0.39
1992	0.42	2.72	0.94	2.79	-0.27	1.22	-0.09	0.85	0.33	1.33	1.35	1.36
1991	3.01	1.40	0.52	1.32	1.82	0.30	1.98	1.00	0.73	2.75	0.01	1.56
1990												2.77

If these returns were in fact achievable they would dominate those obtained from investing directly in the S&P 500 Index for instance. In fact investing in the index offers comparable returns with much higher volatility. Figure 1 compares the Fairfield Sentry (FS) performance with the strategy of investing directly in the S&P 500 with dividends reinvested over the period December 1990 to October 2008. It shows how an initial investment of one hundred would have grown under both assumptions. One hundred invested in the FS fund would have accumulated to 603.8 by October 2008 whereas one hundred invested in the S&P would have accumulated to 433.03 by October 2008 reflecting a lower growth rate. The annual return from investing in the S&P has been 9.64% with a standard deviation of 14.28% over the 17 years and eleven months period. We note that we assume no expenses in the S&P investment and it is not clear if the Fairfield Sentry returns are net[3] of expenses.

We note that the growth of the investment in Fairfield Sentry is approximately linear.

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[3] It is conventional for hedge fund returns to be quoted net of expenses. These expenses include a fee on the total assets plus an incentive fee based on the performance of the fund. See Gregoriou and Lhabitant (2009) as well as Markopolos (2009) and SEC (2005) for more discussion on the unusual fee structure of Madoff's fund.

We also explore this issue in Figure 1. The top green line represents a linear increase of 2.343 per month starting at 100 in December 1990 to October 2008. The bottom green line assumes that a constant monthly compounded[4] growth rate of 84 basis points per year. This assumes that the fund increases at a constant compound rate of return each month. We note that the actual performance of the FS Fund lies inside these two green lines.

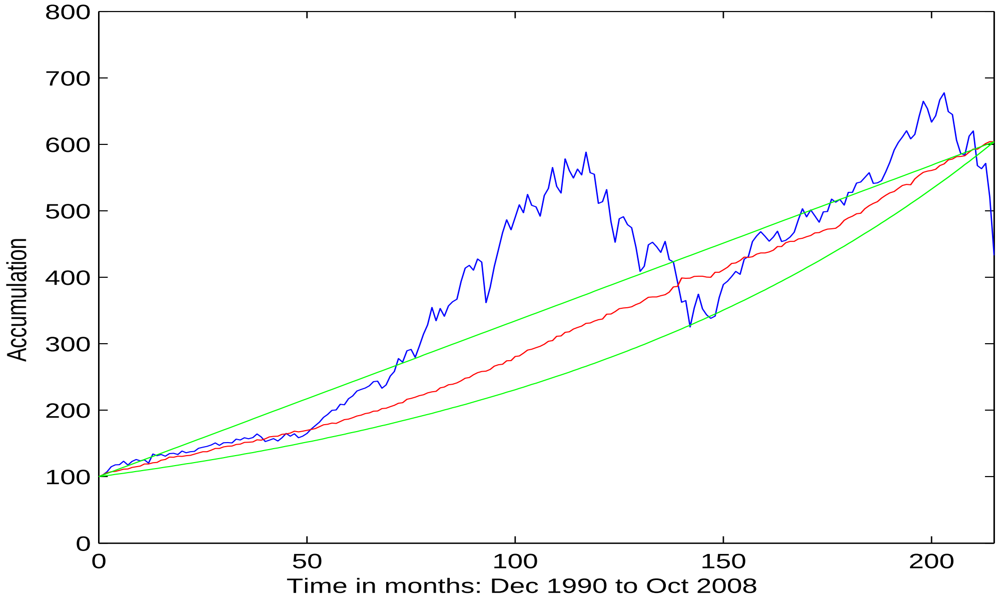

Figure 1: Accumulated investment proceeds using the Fairfield Sentry returns given in Table 1 and investment in the S&P 500 with dividends reinvested.

The Fairfield Sentry performance is in red and the S&P 500 accumulation is in blue. The top green line shows how the initial 100 accumulates to 603.78 in October 2008 if the dollar increase is constant and equal to 2.343 per month. The bottom green line shows how the initial 100 in December 1990 accumulates to 603.78 in October 2008 under a constant monthly compounded growth rate.

Given the consistently high returns and the incredibly low volatility of the FS returns it is of interest to examine what sort of returns could be expected under a split-strike strategy during this period. To do so, we assume that a hypothetical investor takes a long position in the S&P 500 Index starting in December 1990. At the same time he buys a put option on the

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[4] This rate is actually .008398 and it corresponds to a geometric growth rate per month. Thus we have $(1.008398)^{215} = 6.0377$.

index and sells short a call option on the S&P 500 Index[5]. For the purpose of illustration, we assume that the strike price of the put is 5% below the initial spot price of the index and that the strike price of the call is 5% above the initial spot price of the index[6]. Both options are assumed to be European and have a one month maturity. Typically the call price will exceed the put price and the proceeds are invested for one month at the risk-free rate. At the end of the month the option positions are settled in cash. If there is a loss under the option strategy it is financed in the first place from the risk-free investment of the net option premiums and if that is not enough by selling enough shares of the index.

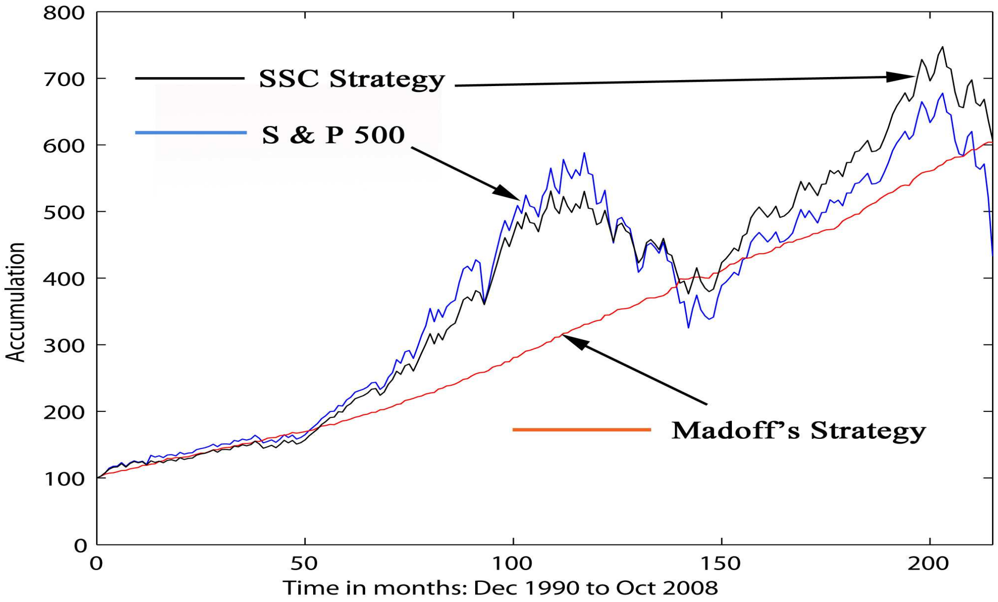

Figure 2: Accumulated investment proceeds using the Fairfield Sentry returns given in Table 1 and investment in the S&P 500 with dividends reinvested.

The Fairfield Sentry performance is in red and the S&P 500 accumulation is in blue. The black line shows the results of investing in the split-strike strategy when the options are priced using the prevailing value of the VIX.

It is assumed that all available monies are invested in the index at the start of the second

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[5] We compare Madoff's strategy with a split-strike strategy on the S&P 500. The FS prospectus refers to the S&P 100. However results of the empirical study would have been similar or even worse. Indeed Gregoriou and Lhabitant (2009) note that "it would have been prohibitively expensive using S&P 100 Index options" since they "are much less widely used than S&P 500 Index options".

[6] Other examples of split-strike strategies with alternative strikes have been investigated by Clauss, Roncalli and Weisang (2009) but their conclusions are similar. The impact of the choice of the strikes in a split-strike conversion strategy is also studied in section 3 from a theoretical perspective.

month and the same option strategy is implemented. This procedure is repeated every month for 215 months. We ignore transaction costs and any price impact of the trades. In order to price the options we use the Black-Scholes formula. As a proxy for the implied volatility we use the VIX to price both[7] the call and put options. The average level of the VIX over this period is 19.24%. We use the prevailing one month US T-bill rates to proxy the risk-free rates in pricing the options.

Table 2: **Summary performance statistics for four strategies for the period December 1990 to October 2008.**

Strategy	Invest in S&P	Split-strike no volatility skew	Split-strike with volatility skew	Fairfield Sentry
Average return (monthly)	0.77	0.92	0.63	0.84
Average return (annual)	9.64	11.68	7.89	10.59
St deviation (monthly)	4.12%	3.09%	3.15%	0.71%
St deviation (annual)	14.28%	10.72%	10.91%	2.45%
Sharpe Ratio (monthly)	.105	.180	.094	.712
Sharpe Ratio (annual)	.363	.656	.326	2.47
Max monthly return	11.44	5.37	4.94	3.29
Min monthly return	-16.79	-4.92	-5.63	-0.64
Percent positive	64.65%	64.65%	64.19%	92.33%
Correlation with $S\&P$	1.0000	0.9480	0.9514	0.3197

The results are summarized in Figure 2 and Table 2. Figure 2 shows that the split-strike conversion strategy (black curve) appears to do quite well as compared to direct investing in the S&P 500 Index (blue curve). Table 2 gives the performance statistics of both strategies. The split-strike conversion strategy has a higher expected return than the FS strategy (11.68% as against 10.59%). However the returns are more highgly correlated with S&P Index and have much higher volatility than the FS strategy.

As shown in Table 2 the expected return for the split-strike strategy is 11.68% per annum with a standard deviation of 10.72% leading to an annual Sharpe ratio[8] of 0.656. While this Sharpe ratio is much less than the FS Sharpe ratio of 2.47 it is almost twice the Sharpe ratio of investing in the index over this period. Our later analysis in Section 3 will show that

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[7] We neglect the change that occurred in 2003 in the calculation of the VIX. The original VIX was based on S&P 100 Index option prices whereas the new VIX uses options on the S&P 500 Index. In addition the VIX Index is only an approximation of the implied volatility of an at-the-money call option. Later we incorporate a volatility skew into the price calculations.

[8] Sharpe (1964,1966,1994).

the split-strike conversion cannot produce a Sharpe ratio twice as big as direct investing in the index. As we will discuss later, there is a maximum possible Sharpe ratio that could be achieved (Goetzmann et al. (2002)). So this result is very surprising. Another way could be to compare the variance and the expected return of Madoff to other stocks and hedge funds. Clauss et al. (2009) show that all funds that have a significant investment in Madoff's fund lie outside the efficiency frontier of the CAPM (see Figure 2 of Clauss et al. (2009)).

The option prices that were used to construct the black curve in Figure 2 or 3 assume that both the call and put options were priced using the prevailing value of the VIX.

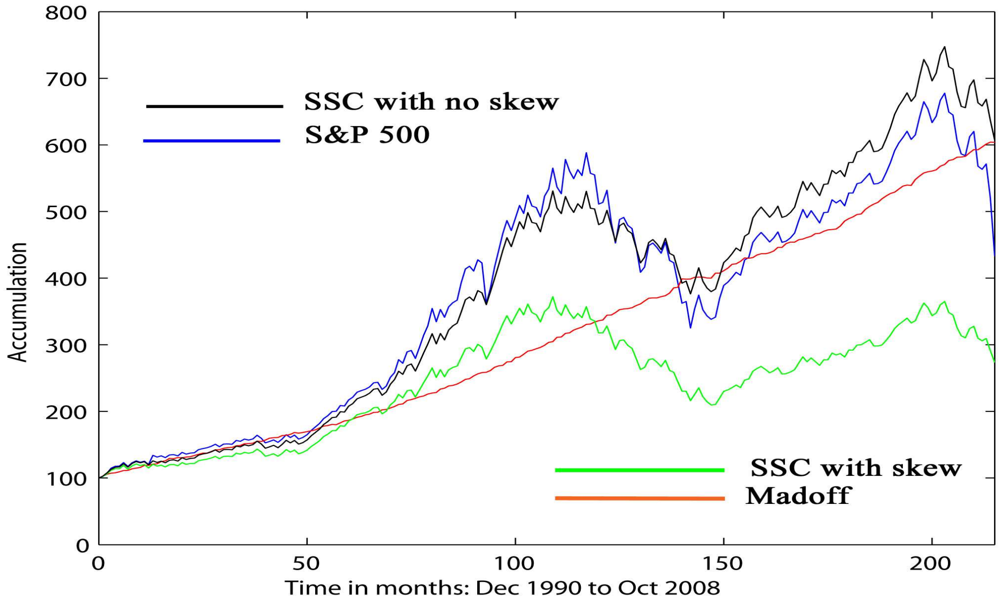

Figure 3: Accumulated investment proceeds using the Fairfield Sentry returns given in Table 1 and investment in the S&P 500 with dividends reinvested.

The Fairfield Sentry performance is in red and the S&P 500 accumulation is in blue. The black line shows the results of investing in the split-strike strategy when the options are priced using the prevailing value of the VIX. The green line shows the results of investing in the Split-Strike Conversion strategy when the options are priced using a volatility skew assumption based on data kindly supplied by Prof. Gurdip Bakshi.

It is well known that there is a volatility skew whereby the implied volatility at a fixed maturity is a decreasing function of the strike price. For example Zhang and Shu (2004) find that over the five year period from 1995 to 1999 the average implied volatility at the money for short term options is 19% whereas the average implied volatility for out-of-the money calls is 17% and the average implied volatility for out-of-the money puts is 21%. To obtain the Figure 3, we use data for the implied volatility skew over 1990 to 2008.

Looking at the impact of the skew complements the study of Clauss et al. (2009). These authors show that it is possible to construct a split-strike strategy with a very low volatility but then it also has a very low return (to do so, one needs to buy put options almost at-the-money). Then, they argue that the only way to obtain a similar trend of the returns as Madoff's returns is to assume that Madoff was an outstanding stock-picker. Ignoring the skew but including an 8.5% extra return per year, Clauss et al. (2009) construct a split-strike strategy that gives similar returns as Madoff (see Figure 5 of Clauss et al. (2009). Including the impact of the skew on the strategy's cost in their study would lead to a much higher extra return than 8.5%. Note that if Madoff was indeed able to generate an 8.5% additional return by picking the right stock, then buying so much protection as he claimed would have been unnecessary.

Manipulation Proof Performance Measures

Recently Goetzmann, Ingersoll, Spiegel and Welch (2007) (GISW) have developed a manipulation free portfolio performance measure. GISW demonstrate that their measure is robust to various manipulation strategies. Even though their measure was not designed to detect outright fraud it can provide valuable insights on the nature of the split-strike strategy and the Fairfield Sentry returns. The formula for the GISW measure $\hat{\Theta}$ for a series of N monthly returns is defined as follows

$$\hat{\Theta} = \frac{1}{(1-\rho)h} \ln \left(\frac{1}{N} \sum_{i=1}^{N} \left[\frac{1+r_{pi}}{1+r_{fi}} \right]^{1-\rho} \right) \tag{1}$$

where

- r_{pi} is the rate of return on the portfolio for month i ,
- r_{fi} is the risk-free rate for month i ,
- h is the time interval in years. Here $h = \frac{1}{12}$.
- ρ corresponds to the relative risk aversion of the investor.

GISW note that this measure $\hat{\Theta}$ has an intuitive economic interpretation. It measures the portfolio's implied excess return after adjusting for risk. Thus for the risk-free portfolio,

$\hat{\Theta} = 0$. If one had a portfolio that earned exactly fifty basis points above the risk-free rate every month with no variation this portfolio would have $\hat{\Theta} = .06$. If the portfolio is risky then $\hat{\Theta}$ decreases if a more risk-averse investor is considered. Table 3 shows the values of the GISW measure for direct investment in the S&P, the split-strike strategy (incorporating the volatility skew) and the Fairfield Sentry returns for different levels of ρ.

Table 3: **Values of $\hat{\Theta}$ corresponding to different levels of ρ for three investment strategies**

Value of ρ	Invest in S&P	Split-strike with volatility skew	Fairfield Sentry
2	0.0309	0.0237	0.0597
3	0.0201	0.0177	0.0594
4	0.0090	0.0117	0.0592
5	-0.0025	0.0056	0.0589
10	-0.0664	-0.0254	0.0575

Table 3 shows that the Fairfield Sentry portfolio outperforms the other strategies based on this measure. If the investor becomes more risk averse the value of $\hat{\Theta}$ declines rapidly for the investment in the $S\&P$ and the split-strike strategy. However $\hat{\Theta}$ hardly changes at all for the Fairfield Sentry returns. The Fairfield Sentry returns correspond to an extra six percent per year above the risk-free rate for all investors even the most risk averse. Mr Madoff's returns were ingeniously designed to appeal to even the most risk averse investors.

This section showed that returns on Fairfield Sentry portfolio looked very suspicious. There is empirical evidence that volatility was too low, that the Sharpe ratios were too high to be true. However they were designed such that any investors would be willing to receive these returns even the most risk averse. Investors chose to invest with B. Madoff because they had full confidence in B. Madoff. He was a former chairman of the NASDAQ. *"His solid and consistent track record generated a mixture of amazement, fascination, and curiosity. Investing with him was an exclusive privilege (...) All Madoff investors should in retrospect kick themselves for not asking more questions before investing. As many of them have learned there is no substitute for due diligence (...) there were a number of red flags in Madoff's investment advisory business that should have been identified as serious concerns and warded off potential clients."* (Gregoriou and Lhabitant (2009)). The next section supplements this empirical study by providing some formal analysis of what the returns under a split-strike conversion should be in the well-accepted model of Black and Scholes. This theoretical study confirms the previous analysis and shows also that even a simple model could help to identify a fraud and detect unrealistic returns.

3 Theoretical Analysis of Split-Strike Strategy

The following section is a theoretical analysis of the split-strike strategy. It is implemented in the standard Black-Scholes framework. The goal of this section is to provide some theoretical support to the suspicion that Fairfield Sentry portfolio was committing a fraud that could have been detected much earlier.

The market is assumed to be arbitrage-free. Let S_0 be the price of the underlying index at current time zero. Assume the index pays no dividends and follows a geometric Brownian motion under the real world measure P so that

$$d\,S_t = \mu\,S_t\,dt + \sigma\,S_t\,d\,W_t \tag{2}$$

where W_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the filtration $\{\mathcal{F}_{\mathrm{t}}\}$, and μ and σ are constants. The risk-free rate r is constant and continuously compounded. The index value at time h writes as $S_h = S_0 \exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)h + \sigma W_h\Big)$. It follows a lognormal distribution. Its two first moments are given as follows,

$$\begin{cases} \mathbb{E}_P[S_h] = S_0 e^{\mu h} \\ \mathbb{V}\mathrm{ar}_P[S_h] = S_0^2 e^{2\mu h}\left(e^{\sigma^2 h} - 1\right) \end{cases} \tag{3}$$

We first describe the split-strike strategy, then derive the first two moments of the standard call and put options and finally obtain some theoretical results about the expected return, the standard deviation, the correlation and the Sharpe ratio of the split-strike strategy.

3.1 Split-Strike Strategy

Suppose the time horizon is h. At time zero, the portfolio manager buys one share of the index and simultaneously sells a call option at a premium c_0 and buys a put option at the premium p_0. The call option has a strike price K_c, the put option has a strike price K_p where

$$K_c > S_0 > K_p.$$

Both options have the same time to maturity $T \geq h$ and are priced by the Black-Scholes formula. The time zero value of the portfolio V_0 is

$$V_0 = S_0 = S_0 + (c_0 - p_0) \ + \ (\textit{long put} \ + \ \textit{short call}) \tag{4}$$

We assume that the amount $(c_0 - p_0)$ accumulates to the end of the period at the risk-free rate. The value of the call option (respectively the put option) at time h if the stock price is S_h is denoted by $\mathcal{C}_h$ (respectively $\mathcal{P}_h$). The value of the portfolio at time h is

$$V_h = S_h + (c_0 - p_0)e^{rh} + (\mathcal{P}_h - \mathcal{C}_h). \tag{5}$$

Remark 1 When $K_c = K_p = K$, the payoff of the strategy is deterministic and equal to

$$V_h = S_0 e^{rh} \tag{6}$$

This is a straightforward consequence of the call-put parity. Indeed, the call-put parity relationship applied at time h implies $V_h = Ke^{-r(T-h)} + (c_0 - p_0)e^{rh}$. But at time 0, the call-put parity can be written as $c_0 - p_0 = S_0 - Ke^{-rT}$. Thus (6) is proved.

We are interested in the general case when K_p the strike of the put option is different from the strike K_c of the call option. The Sharpe ratio of this portfolio is

$$\mathcal{SR}\left(V_h\right) = \frac{\frac{\mathbb{E}_P[V_h]}{V_0} - e^{rh}}{\sqrt{\frac{\mathbb{V}\mathrm{ar}_P[V_h]}{V_0^2}}} \ = \ \frac{\mathbb{E}_P[V_h] \ - \ S_0 e^{rh}}{\sqrt{\mathbb{V}\mathrm{ar}_P[V_h]}} \tag{7}$$

since $V_0 = S_0$. The expected value of the portfolio and its standard deviation are respectively calculated as follows

$$\begin{cases} \mathbb{E}_P[V_h] = \mathbb{E}_P[S_h] + \mathbb{E}_P[\mathcal{P}_h] - \mathbb{E}_P[\mathcal{C}_h] + (c_0 - p_0)e^{rh} \\ \mathbb{V}\mathrm{ar}_P[V_h] = \mathbb{V}\mathrm{ar}_P[S_h] + \mathbb{V}\mathrm{ar}_P[\mathcal{P}_h] + \mathbb{V}\mathrm{ar}_P[\mathcal{C}_h] + 2\left(\mathbb{C}\mathrm{ov}_P(S_h, \mathcal{P}_h) - \mathbb{C}\mathrm{ov}_P(S_h, \mathcal{C}_h) - \mathbb{C}\mathrm{ov}_P(\mathcal{C}_h, \mathcal{P}_h)\right) \end{cases}$$

To derive expressions of the Sharpe ratio, the expected return and the variance of the split-strike strategy, we need to know the moments of standard options. The next paragraph gives the expressions of the first moment of standard options. Second moments are provided in the appendix.

3.2 First moments of standard options under the physical measure

The price dynamics of the underlying asset S under the P measure are given by (2). Denote by X_T the payoff of the option (in the case of the call option $X_T = \max(S_T - K_c, 0)$ and in the case of the put $X_T = \max(K_p - S_T, 0)$). Let h be such that $0 < h < T$. Denote by X_h the value of the derivative at time h.

Let us denote by $\mathcal{C}_h$ and $\mathcal{P}_h$ the value at time h of respectively the call option and the put option in the Black-Scholes framework. The price is expressed at time h with current asset price S_h at time h, with respective exercise prices K_c and K_p and maturity T.

$$\mathcal{C}_h := \mathcal{C}[S_h, h, K_c, T], \quad \mathcal{P}_h := \mathcal{P}[S_h, h, K_p, T].$$

Proposition 3.1. *First moments of standard options:*
The first moments of this call option and this put option are respectively given as follows

$$\mathbb{E}_P\left[\mathcal{C}_h\right] = \mathcal{C}\left[S_0 e^{\mu h}, 0, K_c e^{rh}, T\right] = S_0 e^{\mu h}\,\Phi\left(\widetilde{d}_1(K_c)\right) - K_c e^{rh}\,e^{-rT}\Phi\left(\widetilde{d}_2(K_c)\right), \tag{8}$$

$$\mathbb{E}_P\left[\mathcal{P}_h\right] = \mathcal{P}\left[S_0 e^{\mu h}, 0, K_p e^{rh}, T\right] = K_p e^{rh}\,e^{-rT}\Phi\left(-\widetilde{d}_2(K_p)\right) - S_0 e^{\mu h}\,\Phi\left(-\widetilde{d}_1(K_p)\right), \tag{9}$$

where

$$\widetilde{d}_1(K) = \frac{\ln\left(\frac{S_0 e^{\mu h}}{K e^{rh}}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad ; \quad \widetilde{d}_2(K) = \widetilde{d}_1(K) - \sigma\sqrt{T}$$

and Φ is the cdf of a standard normal distribution $\mathcal{N}$(0,1).

It turns out that in the case of standard call and put options, explicit formulae for their first and second moments are available in Cox and Rubinstein (1985)[9]. A full proof of the proposition is provided in the appendix.

The first moment of the distribution of the call price has a simple and intuitive form. Note that the resulting expression is equal to a Black-Scholes call option with the same time to maturity as the initial call and the same volatility and interest rate. However it has a higher input asset price and a higher input strike price. The new input asset price is equal to the expected value (under P) of the asset price at time h, $\mathbb{E}_P[S_h] = S_0 e^{\mu h}$. The new input strike price is equal to the original strike price, respectively K_c or K_p, accumulated at the risk-free rate.

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[9] Chapter 6, footnote 34 for the first moments and footnote 43 for the covariance between two calls.

The limit cases when $h = 0$ or $h = T$ are easily verified. When $h = 0$, the result is well-known. When $h = T$, the result can be found in the appendix of Goetzmann et al. (2002,2007).

3.3 Properties of the Split-Strike Strategy

In this section, we present a number of useful results concerning the split-strike conversion strategy. To derive these results, we use formulae for the moments of the option prices given in the previous section or in the appendix. The first proposition related to the expected return under the Split-Strike Conversion strategy.

Proposition 3.2. *Expectation of the strategy:*
The expected payoff of the strategy is equal to

$$\mathbb{E}_P\left[V_h\right] = S_0e^{\mu h} - \mathcal{C}\left[S_0e^{\mu h}, 0, K_ce^{rh}, T\right] + \mathcal{P}\left[S_0e^{\mu h}, 0, K_pe^{rh}, T\right] + (c_0 - p_0)e^{rh}.$$

When $K_c = K_p = K$, V_h is deterministic and its expectation is equal to $\mathbb{E}_P\left[V_h\right] = S_0e^{rh}$.

The result is a consequence of Proposition 3.1. The special case when $K_c = K_p = K$ is established earlier. See expression (6). ☐

Remark 2 The expected return from the strategy is an increasing function of K_c and an increasing function of K_p.

Remark 3 The split-strike strategy has a lower expected return and a lower variance than a direct investment in the index[10]:

$$\mathbb{E}_P[V_h] \leqslant \mathbb{E}_P[S_h] \ \ \textit{and} \ \ \mathbb{V}\mathrm{ar}_P[V_h] \leqslant \mathbb{V}\mathrm{ar}_P[S_h].$$

Proposition 3.3. *Variance and Sharpe ratio of the strategy:*
The variance of the strategy at time h is equal to

$$\mathbb{V}\mathrm{ar}_P\left[V_h\right] \ = \ S_0^2e^{2\mu h}\left(e^{\sigma^2 h} - 1\right) + \mathbb{V}_{\mathcal{C}} + \mathbb{V}_{\mathcal{P}} - 2\mathbb{C}ov_{\mathcal{C},\mathcal{P}} - 2\mathbb{C}ov_{\mathcal{C},\mathcal{S}} + 2\mathbb{C}ov_{\mathcal{S},\mathcal{P}}$$

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[10] Proof available from the authors upon request.

where

$$\begin{cases} \mathbb{V}_{\mathcal{C}} = \mathbb{E}_P\left[\mathcal{C}_h^2\right] - \mathbb{E}_P\left[\mathcal{C}_h\right]^2 \\ \mathbb{V}_{\mathcal{P}} = \mathbb{E}_P\left[\mathcal{P}_h^2\right] - \mathbb{E}_P\left[\mathcal{P}_h\right]^2 \\ \mathbb{C}ov_{\mathcal{C},\mathcal{P}} = \mathbb{E}_P\left[\mathcal{C}_h\mathcal{P}_h\right] - \mathbb{E}_P\left[\mathcal{C}_h\right]\mathbb{E}_P\left[\mathcal{C}_h\right] \\ \mathbb{C}ov_{\mathcal{C},\mathcal{S}} = \mathbb{E}_P\left[\mathcal{C}_h S_h\right] - \mathbb{E}_P\left[\mathcal{C}_h\right] S_0 e^{\mu h} \\ \mathbb{C}ov_{\mathcal{P},\mathcal{S}} = \mathbb{E}_P\left[\mathcal{P}_h S_h\right] - \mathbb{E}_P\left[\mathcal{P}_h\right] S_0 e^{\mu h} \end{cases}$$

First moments can be found in Proposition 3.1 and formulae for second moments are in the appendix A (see (18) *and* (19)*) and the 3 cross products:* $\mathbb{E}_P\left[\mathcal{C}_h\mathcal{P}_h\right], \mathbb{E}_P\left[\mathcal{C}_h S_h\right], \mathbb{E}_P\left[\mathcal{P}_h S_h\right]$ *can be found in the Appendix B.*

The Sharpe ratio of the strategy is defined by the equation (7)*:*

$$\mathcal{SR}\left(V_h\right) = \frac{\mathbb{E}_P[V_h] - S_0 e^{rh}}{\sqrt{\mathbb{V}\mathrm{ar}_P[V_h]}}$$

where the expectation is given in proposition 3.2.

The Sharpe ratio is not defined when $K_p = K_c$ because both the numerator and the denominator are equal to zero. This fact will be explained in the numerical analysis in the following section.

Proposition 3.4. *Correlation of the strategy with the index* S*:*
The correlation can be written as

$$Corr(V_h, S_h) = \frac{\mathbb{V}\mathrm{ar}_P[S_h] - \mathbb{C}ov_{\mathcal{C},\mathcal{S}} + \mathbb{C}ov_{\mathcal{P},\mathcal{S}}}{\sqrt{\mathbb{V}\mathrm{ar}_P[V_h]}\, S_0 e^{\mu h}\sqrt{e^{\sigma^2 h} - 1}}$$

where all terms are given in Prop. 3.3.

The correlation of the strategy with the index is of course equal to 0 when $K_c = K_p$, but similar to the Sharpe ratio, it is not defined at zero.

4 Numerical Analysis

Consider a split-strike strategy with maturity $T = h$. Similar results hold when $h < T$. We consider two cases. First, we assume the strikes of respectively the call option and the put option are given by

$$K_c = S_0 + b \quad K_p = S_0 - b \tag{10}$$

where $b \in [0, S_0]$. Second, we study the case when K_p and K_c are chosen independently. We will finally discuss optimal choices of the two strikes.

4.1 Case when the strikes are $K_c = S_0 + b,\ K_p = S_0 - b$ with $b \in [0, S_0]$

We plot the expected payoff of the strategy and the standard deviation when b varies in Figure 4. We assume plausible values for the parameters of the financial market. The conclusions hold for other choices of the volatility σ, the interest rate r, the instantaneous expected return μ and maturity of the strategy T. Note that it is not necessary that $h = T$.

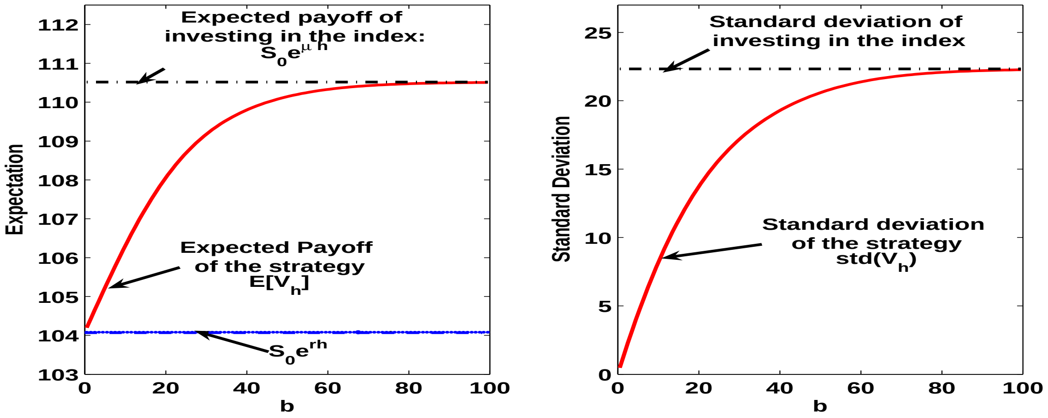

Figure 4: Expectation and standard deviation of the split-strike strategy. Assume $S_0 = 100$, $\sigma = 20\%$, $\mu = 0.1$ and $r = 0.04$. Assume $h = 1, T = 1$. The strikes of the call and the put are $K_c = S_0 + b$ and $K_p = S_0 - b$. On the left panel, we display the expected payoff of the strategy and of an investment in the index ($\mathbb{E}_P[V(h)]$ and $\mathbb{E}_P[S_h]$). The right panel represents the standard deviations $std[V(h)]$ and $std[S_h]$. The range of b is [0,100].

Consistent with our theoretical findings, Figure 4 shows that the expected return of a split-strike strategy is always lower than the expected return of investing in the index. A lower expected return is compensated by a lower standard deviation. Note that as b goes to 0, the expected payoff of the strategy converges to S_0e^{rh} which means that the return of the strategy is the risk-free rate. This is not a surprise because when $K_c = K_p = S_0$, $V(h) = S_0e^{rh}$. In this case $V(h)$ is deterministic and its variance is equal to 0.

Figure 5 displays the Sharpe ratio of the strategy against the Sharpe ratio of investing in the index with a horizon $h = T = 1$ year. We observe that the positions in options can enhance the Sharpe ratio. However the enhancement is bounded from below as well as from above. Goetzmann et al. (2002) show that there is a maximum possible Sharpe ratio attainable in the complete market of Black-Scholes. The formula for this maximum Sharpe ratio when $h = T$ is

$$\sqrt{e^{\frac{(\mu-r)^2 T}{\sigma^2}} - 1}.$$

In addition the limit[11] of the Sharpe ratio when $b \to 0$,

$$\lim_{b\to 0^+} \mathcal{SR}\left(V_T\right) = \frac{\Phi(a_2) - \Phi(\hat{a}_2)}{\sqrt{\Phi(a_2)(1-\Phi(a_2))}}$$

where $a_2 = \frac{\mu\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$, $\hat{a}_2 = \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$. When $b = 0$, the strategy is equivalent to investing in bonds and the Sharpe ratio is not defined.

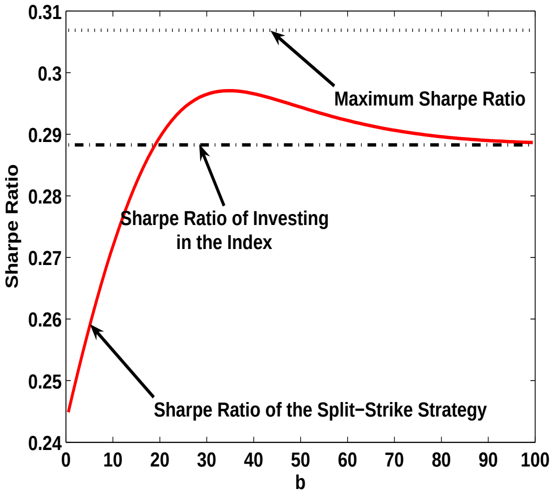

Figure 5: Sharpe ratio of the split-strike conversion strategy compared to the Sharpe ratio of investing in the index.

Here $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = 1, T = 1$. The strikes of the call and the put are respectively equal to $K_c = S_0 + b$ and $K_p = S_0 - b$, $b \in [0, 100]$.

Figure 5 assumes $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = T = 1$ year. In this case, the minimum Sharpe ratio is 0.2432 and the maximum Sharpe ratio is equal to 0.3069. Note also that Figure 5 is consistent with Figure 3 of Lhabitant (1998).

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[11] Proof available from the authors upon request.

Finally, in Figure 6 we investigate the correlation between the strategy and the index and the beta of the strategy. We assume the underlying index S is a good proxy for the financial market. Then the beta is defined as follows:

$$\beta = \frac{\mathbb{C}\text{ov}_P\left(\frac{V_h}{V_0}, \frac{S_h}{S_0}\right)}{\mathbb{V}\text{ar}_P\left(\frac{S_h}{S_0}\right)} = \frac{\mathbb{C}\text{ov}_P(V_h, S_h)}{\mathbb{V}\text{ar}_P(S_h)}$$

Formulas for the covariance of V_h and S_h are established in Proposition 3.3 and explicitly given in the Appendix B.

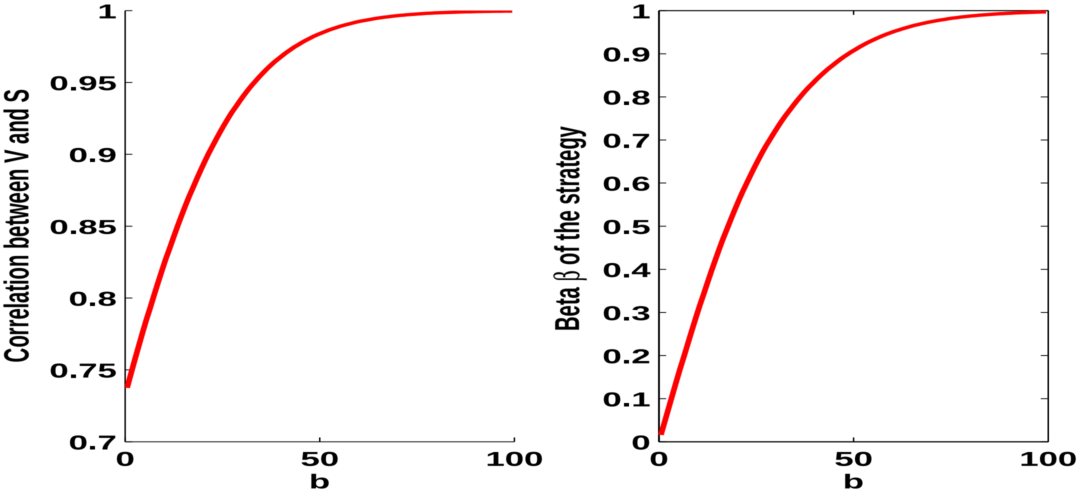

Figure 6: Correlation between S_h and V_h and Beta of the strategy.
Here $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = T = 1$. The strikes of the call and the put are respectively equal to $K_c = S_0 + b$ and $K_p = S_0 - b$. $b \in [0, 100]$.

Figure 6 shows that the strategy is highly correlated with the index and that the beta lies between zero and one. When $b = 0$, the strategy is deterministic and the correlation is not defined but the beta is defined and equal to 0. Similar to the Sharpe ratio, the limit of the correlation when $b \longrightarrow 0^+$ is positive

$$\lim_{b \to 0^+} Corr\,(V_T, S_T) = \frac{\Phi(a_1) - \Phi(a_2)}{\sqrt{\Phi(a_2)(1 - \Phi(a_2))\,(e^{\sigma^2 T} - 1)}}$$

where[12] $a_1 = \frac{\mu\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}$, $a_2 = \frac{\mu\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$. If $b > 0$, the correlation is always greater

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[12] Proof available from the authors upon request.

than 0.732 (which is its limit calculated with the same parameters as before). However, as reported to the SEC in 2005 by H. Markopolos, the beta of the strategy was 0.06 and the correlation with the index was only 0.3 (see attachment 1 on Fairfield Sentry Performance Data in 2005 in SEC(2005)). The formal analysis in this section confirms that the returns claimed by Madoff are theoretically impossible.

4.2 General case

Consider now the case of the split-strike strategy where the strike prices of the call and the put, respectively K_c and K_p vary independently. Similar results as before can be obtained for the expectation, the variance, the correlation and the beta of the strategy. We only present the Sharpe ratio of the strategy in Figure 7.

Figure 7 shows that there are choices of strikes for the call and for the put that maximize the Sharpe ratio. The optimal strikes do not necessarily correspond to the symmetric case with respect to S_0 as we will see.

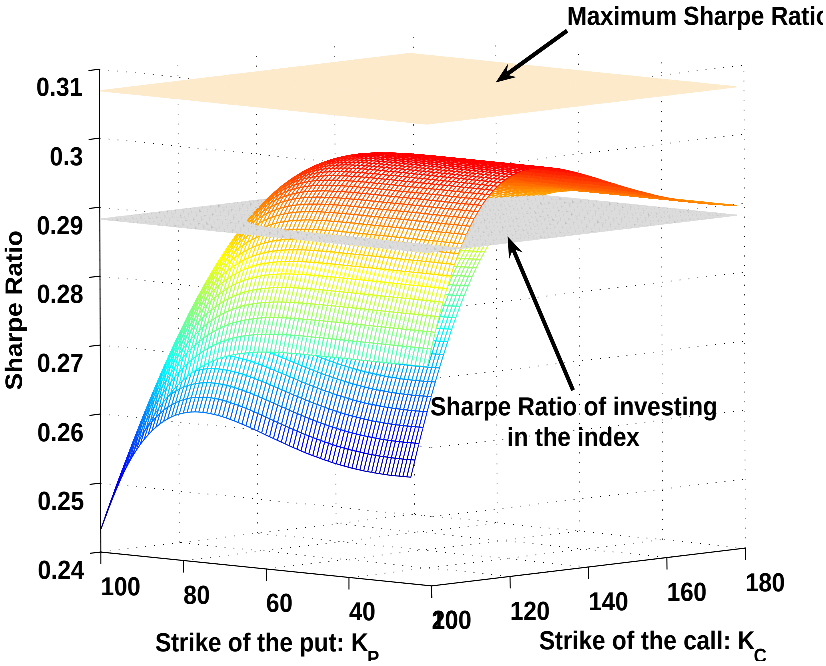


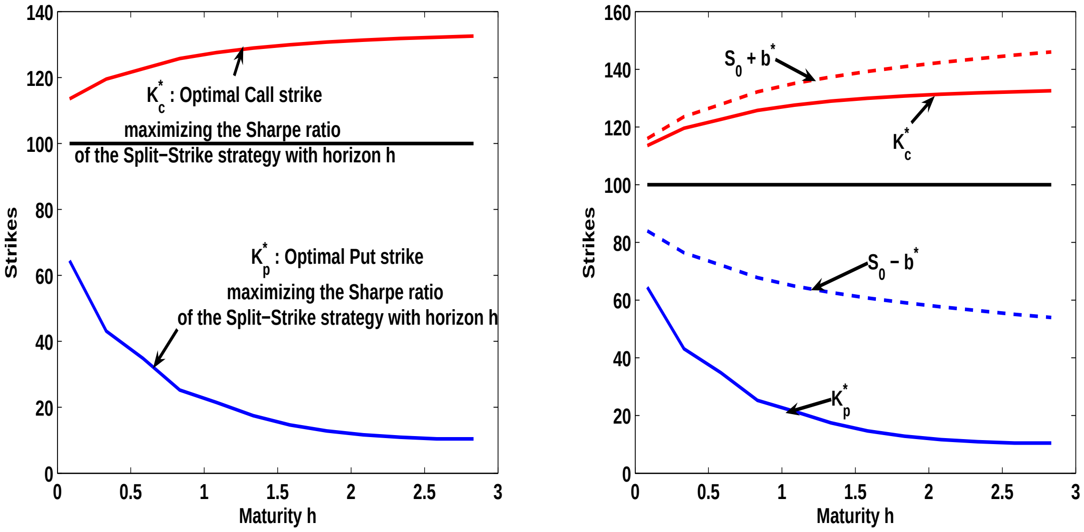
Figure 7: Sharpe ratio of the strategy versus Sharpe ratio of investing in the index. Here $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = T = 1$. The strikes of the call and the put are respectively equal to K_c and K_p.

4.3 Optimal choice of the parameters K_p and K_c

In this last subsection, we numerically derive the optimal strikes of the put and of the call in a split-strike strategy for different choices of the horizon $h = T$. The objective is to maximize the Sharpe ratio over the given maturity.

Figure 8: Choice of the strikes K_c and K_p to maximize the Sharpe ratio

Here $\sigma = 20\%$, $\mu = 10\%$ and $r = 4\%$. The strikes of the call and the put are respectively equal to K_c and K_p. On the left panel, the optimal strikes K_c^* and K_p^* are determined in the general case. On the right panel, the optima are compared with the ones obtained by maximizing the Sharpe ratio when strikes are symmetric with respect to S_0. In this case, the optima are denoted by $K_c = S_0 + b^*$ and $K_p = S_0 - b^*$



As mentioned in the literature by Goetzmann et al. (2007) and by Cvitanic, Lazrak and Wang (2009), the choice of the horizon can dramatically change the results. With a longer horizon, it is optimal to buy a call with a higher strike and a put with a lower strike. Note also that it is optimal not to have a perfectly symmetric split-strike strategy. It is optimal to buy a put option more deeply out-of-the-money than the call option.

5 Conclusions

This paper analyzed certain features of Bernie Madoff's investment performance. It is now known that these results were based on a giant Ponzi scheme which flourished for a long time despite several red flags and the highly suspicious nature of the returns. Indeed were it not for the current financial crisis it seems likely that the Madoff investment scheme would still be in operation.

We implemented a version of the split-strike strategy similar to the one allegedly used by Madoff and compared the results with those reported by Fairfield Sentry one of Madoff's feeder funds. The Sharpe ratio based on our version of the split strike strategy was very much lower than Fairfield Sentry's Sharpe ratio over the same period. In addition the correlation between the split-strike strategy and the market was more than twice the corresponding correlation for the Madoff strategy. One of the most unbelievable statistics of Madoff's performance is the very low volatility. This makes the Madoff's returns very attractive even to the most risk averse investors. These returns were concocted in a very clever way.

Our theoretical analysis reaches the same conclusions. There are closed-form expressions for the moments of the split strike strategy and its correlation with the market. In addition there is a theoretical maximum Sharpe ratio that can be obtained using options. We find that the performance statistics reported by Fairfield Sentry lie well outside their theoretical bounds. These results are incredible in the most literal sense.

In summary there are some simple quantitative diagnostics that should have raised suspicions about Madoff's performance.

Appendix

Let us first recall some of the key properties of the normal distribution and of the bivariate normal distribution that will be useful to derive a closed-form expression of the Sharpe ratio of the strategy under study. Consider $X \sim \mathcal{N}\left(\mu_x, \sigma_x^2\right)$ and denote by $g\left(x, \mu_x, \sigma_x^2\right)$ the corresponding pdf. Let $c \in \mathbb{R}$. The pdf of X satisfies

$$g\left(x, \mu_x, \sigma_x^2\right) e^{cx} \;=\; g(x, \mu_x + c\sigma_x^2, \sigma_x^2)\; e^{\mu_x c + \frac{c^2\sigma_x^2}{2}}, \tag{11}$$

or in other words, if f_X denotes the pdf of X then

$$f_X\left(x\right) e^{cx} \;=\; f_{X + c\sigma_x^2}(x)\; e^{\mu_x c + \frac{c^2\sigma_x^2}{2}}.$$

Consider (X, Y) following the bivariate distribution

$$(X, Y) \sim \mathcal{N}_2\left(\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho\right),$$

then $X \sim \mathcal{N}\left(\mu_x, \sigma_x^2\right)$ and $Y \sim \mathcal{N}\left(\mu_y, \sigma_y^2\right)$ are two correlated normal random variables with correlation coefficient ρ. The pdf of (X, Y) can be written as

$$f\left(x, \mu_x, \sigma_x^2, y, \mu_y, \sigma_y^2, \rho\right) = \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\left(1-\rho^2\right)} e^{-\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)}.$$

We are going to make use of one particular property of the pdf of the bivariate normal distribution. Given two real numbers α, β, one has

$$e^{\alpha x + \beta y} f\left(x, 0, \sigma_x^2, y, 0, \sigma_y^2, 0\right) = e^{\frac{\alpha^2}{2} + \frac{\beta^2}{2}} f\left(x, \alpha\sigma_x^2, \sigma_x^2, y, \beta\sigma_y^2, \sigma_y^2, 0\right). \tag{12}$$

We will have to correctly choose α and β to calculate the different integrals that will appear later in our calculations.

A Moments of Standard Options

Using standard results, the price at time h of an option can be expressed as a conditional expectation under the risk neutral measure Q defined by its Radon-Nikodym derivative:

$$L_T := \frac{dQ}{dP} = \exp\left\{-\left(\frac{\mu - r}{\sigma}\right) W_T - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} T\right\}. \tag{13}$$

Then from the Girsanov theorem, we know that under the new probability measure Q, $B_t = W_t + \frac{\mu - r}{\sigma} t$ is a standard Brownian motion with respect to $\{\mathcal{F}_t\}$ and the stock price dynamic are expressed as $d\,S_t = r\,S_t\,dt + \sigma\,S_t\,d\,B_t$.

The price of the derivative at time h can be expressed as

$$X_h = e^{-r(T-h)}\mathbb{E}_Q\left[X_T\left|\mathcal{F}_h\right.\right]. \tag{14}$$

We are interested in $\mathbb{E}_P\left[X_h\right]$, the expectation under the physical measure of the value at time h of the strategy. Using equation (14) we can derive a compact expression for the first moment of the distribution as a double expectation with the outer expectation taken over the P-measure and the inner expectation over the Q-measure.

First moment of the call option

The expression we want to calculate is given by

$$\mathbb{E}_P\left[\mathcal{C}_h\right] \;=\; \mathbb{E}_P\left[e^{-r(T-h)}\mathbb{E}_Q\left[\max\left(S_T-K\right)^+\left|\mathcal{F}_h\right.\right]\right].$$

Using Black-Scholes formula, one obtains,

$$\mathbb{E}_P\left[\mathcal{C}_h\right] \;=\; \mathbb{E}_P\left[S_h\Phi\left(d_1\right)-Ke^{-r(T-h)}\,\Phi\left(d_2\right)\right],$$

where

$$d_1 = \frac{\ln\left(\frac{S_h}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}, \quad d_2 = d_1-\sigma\,\sqrt{T-h}.$$

Since W_t is a standard Brownian motion under P, $\frac{W_T-W_h}{\sqrt{T-h}}$ is independent of $\mathcal{F}_h$ and follows a standard normal distribution, one can also write the above formula as follows:

$$\mathbb{E}_P\left[\mathcal{C}_h\right]=\mathbb{E}_P\left[S_h\mathbb{E}_P\left[\mathbb{1}_{\frac{W_T-W_h}{\sqrt{T-h}}<d_1}\left|\mathcal{F}_h\right.\right]\right]-Ke^{-r(T-h)}\mathbb{E}_P\left[\mathbb{E}_P\left[\mathbb{1}_{\frac{W_T-W_h}{\sqrt{T-h}}<d_2}\left|\mathcal{F}_h\right.\right]\right]. \tag{15}$$

However, as we have that $S_h=S_0e^{\mu h-\frac{\sigma^2}{2}h+\sigma W_h}$, we get

$$\begin{aligned} d_1 &= \tfrac{W_h}{\sqrt{T-h}}+\frac{\ln\left(\frac{S_0}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)h+\left(r+\frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}=\tfrac{W_h}{\sqrt{T-h}}+d,\\ d_2 &= \tfrac{W_h}{\sqrt{T-h}}+d-\sigma\sqrt{T-h}, \end{aligned}$$

where

$$d=\frac{\ln\left(\frac{S_0}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)h+\left(r+\frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}.$$

The calculation of (15) can be split into two parts:

- The second part is straightforward:

$$\begin{aligned}\mathbb{E}_P\left[\mathbb{E}_P\left[\mathbb{1}_{\frac{W_T-W_h}{\sqrt{T-h}}<d_2}\,|\,\mathcal{F}_h\right]\right] &= P\left(\frac{W_T-W_h}{\sqrt{T-h}}-\frac{W_h}{\sqrt{T-h}}<d-\sigma\sqrt{T-h}\right)\\ &= \Phi\left((d-\sigma\sqrt{T-h})\sqrt{\frac{T-h}{T}}\right).\end{aligned}$$

  The last equality comes from the fact that $\frac{W_T-W_h}{\sqrt{T-h}}-\frac{W_h}{\sqrt{T-h}}\sim\mathcal{N}\left(0,\sqrt{\frac{T}{T-h}}\right)$.

- For the first part, we can write it as

$$S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)h}\mathbb{E}_P\left[e^{\sigma\sqrt{T-h}\frac{W_h}{\sqrt{T-h}}}\mathbb{E}_P\left[\mathbb{1}_{\frac{W_T-W_h}{\sqrt{T-h}}<d_1}\,|\,\mathcal{F}_h\right]\right] \tag{16}$$

  $\frac{W_T-W_h}{\sqrt{T-h}}$ and $\frac{W_h}{\sqrt{T-h}}$ are two independent normally distributed random variables. Thus the expectation term in (16) can be also written as the following double integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{\sigma\sqrt{T-h}x}\,\mathbb{1}_{y+x<d}\,f\left(x,0,\frac{h}{T-h},y,0,1,0\right)dxdy.$$

  Applying formula (12) with $\alpha=\sigma\sqrt{T-h}$ and $\beta=0$, one obtains

$$e^{\frac{\sigma^2(T-h)}{2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbb{1}_{y+x<d}\,f\left(x,\sigma\frac{h}{\sqrt{T-h}},\frac{h}{T-h},y,0,1,0\right)dxdy,$$

  which is equal to the probability a normally distributed $X\sim\mathcal{N}\left(\sigma\frac{h}{\sqrt{T-h}},\frac{T}{T-h}\right)$ is less than d:

$$e^{\frac{\sigma^2(T-h)}{2}}\Phi\left(\frac{d-\sigma\frac{h}{\sqrt{T-h}}}{\sqrt{\frac{T}{T-h}}}\right).$$

Then one obtains the following (simple) expression:

$$\mathbb{E}_P\left[\mathcal{C}_h\right] = S_0e^{\mu h}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\varepsilon_1}{\sigma\sqrt{T}}\right)-Ke^{-r(T-h)}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\varepsilon_2}{\sigma\sqrt{T}}\right)$$

where $\varepsilon_1=\left(\mu+\frac{\sigma^2}{2}\right)T-(\mu-r)(T-h)$ and $\varepsilon_2=\varepsilon_1-\sigma^2T$. We recognize the expression of the price of a call option:

$$\mathbb{E}_P\left[\mathcal{C}_h\right] = S_0e^{\mu h}\Phi\left(\widetilde{d_1}\right)-(Ke^{rh})e^{-rT}\Phi\left(\widetilde{d_2}\right)=\mathcal{C}\left[S_0e^{\mu h},0,Ke^{rh},T\right],$$

where

$$\widetilde{d}_1 = \frac{\log\left(\frac{S_0 e^{\mu h}}{K e^{rh}}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad \widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{T}. \tag{17}$$

$\square$

The proof of the first moment of the put option is similar and omitted.

Second moment of the call option

Proposition A.1. *Second moments of standard options:*
The second moments of the call and put option are given as follows,

$$\begin{aligned} \mathbb{E}_P\left[\mathcal{C}_h^2\right] &= S_0^2 e^{2\mu h} e^{\sigma^2 h} \Phi_2\left(\widetilde{d}_1(K_c) + \frac{\sigma h}{\sqrt{T}},\ \widetilde{d}_1(K_c) + \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ &\quad -2K e^{-r(T-h)} S_0 e^{\mu h} \Phi_2\left(\widetilde{d}_1(K_c),\ \widetilde{d}_2(K_c) + \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ &\quad +K^2 e^{-2r(T-h)} \Phi_2\left(\widetilde{d}_2(K_c),\ \widetilde{d}_2(K_c),\ \frac{h}{T}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} \mathbb{E}_P\left[\mathcal{P}_h^2\right] &= S_0^2 e^{2\mu h} e^{\sigma^2 h} \Phi_2\left(-\widetilde{d}_1(K_p) - \frac{\sigma h}{\sqrt{T}},\ -\widetilde{d}_1(K_p) - \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ &\quad -2K e^{-r(T-h)} S_0 e^{\mu h} \Phi_2\left(-\widetilde{d}_1(K_p),\ -\widetilde{d}_2(K_p) - \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ &\quad +K^2 e^{-2r(T-h)} \Phi_2\left(-\widetilde{d}_2(K_p),\ -\widetilde{d}_2(K_p),\ \frac{h}{T}\right) \end{aligned} \tag{19}$$

where

$$\widetilde{d}_1(K) = \frac{\ln\left(\frac{S_0 e^{\mu h}}{K e^{rh}}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad ; \quad \widetilde{d}_2(K) = \widetilde{d}_1(K) - \sigma\sqrt{T}$$

and where $\Phi_2(x, y, \rho)$ is the bivariate standard normal distribution function with correlation parameter ρ.

Remark 4 In the case when $h = T$, the result is given in the appendix of Goetzmann et al. (2007). In this case, the result can be simplified and expressed as a combination of the univariate standard normal distribution (since $\Phi_2(a, b, 1) = \Phi(\min(a, b))$).

The Black-Scholes call price expressed at time h with current asset price S_h at time h, with exercise price K and with maturity T is given by

$$\mathcal{C}_h \;=\; S_h\,\Phi(d_1) - K\,e^{-r(T-h)}\Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_h}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}, \quad d_2 = d_1 - \sigma\sqrt{T-h}.$$

We are now looking for an expression of the second moment:

$$\begin{aligned} \mathbb{E}_P\left[\mathcal{C}_h^2\right] &= \mathbb{E}_P\left[S_h^2\Phi(d_1)^2 - 2S_h\Phi(d_1)Ke^{-r(T-h)}\Phi(d_2) + K^2e^{-2r(T-h)}\Phi(d_2)^2\right] \\ &= E_1 - 2Ke^{-r(T-h)}E_2 + K^2e^{-2r(T-h)}E_3. \end{aligned}$$

Let us get an expression for the 3 expectations that appear in the above sum:

$$E_1 := \mathbb{E}_P\left[S_h^2\Phi(d_1)^2\right], \quad E_2 := \mathbb{E}_P\left[S_h\Phi(d_1)\Phi(d_2)\right], \quad E_3 := \mathbb{E}_P\left[\Phi(d_2)^2\right]$$

Computation of the first term E_1:

$$E_1 = S_0^2e^{2\mu h - \sigma^2 h}\mathbb{E}_P\left[e^{2\sigma X}\,\Phi\left(\frac{\sigma X + \ln\left(\frac{S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)h}}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}\right)^2\right]$$

where $X = W_h$ is independent of the σ-field $\mathcal{F}_t$ and follows $\mathcal{N}(0,h)$. Let us define by Y and Z random variables independent of X and of the σ-field $\mathcal{F}_t$ that both follow $\mathcal{N}(0,T-h)$. Denote by

$$k := \frac{1}{\sigma}\ln\left(\frac{S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)h}}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)\frac{(T-h)}{\sigma}.$$

Therefore,

$$E_1 = S_0^2e^{2\mu h-\sigma^2 h}\mathbb{E}_P\left[e^{2\sigma X}\,P\left(Y < X+k\right)P\left(Z < X+k\right)\,\right].$$

This could be written in terms of integrals

$$E_1 = S_0^2e^{2\mu h-\sigma^2 h}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\sigma x}\,\mathbb{1}_{y-x<k}\mathbb{1}_{z-x<k}g(x,0,h)g(y,0,T)g(z,0,T)dxdydz,$$

where $g(x,\mu_x,\sigma_x^2)$ denotes the pdf of $\mathcal{N}(\mu_x,\sigma_x^2)$. Using (11), it could be rewritten as

$$E_1 = S_0^2e^{2\mu h+\sigma^2 h}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbb{1}_{y-x<k}\mathbb{1}_{z-x<k}g(x,2\sigma h,h)g(y,0,T)g(z,0,T)dxdydz.$$

Denote by $\widetilde{X} = X + 2\sigma h$. The above expression could now be written as

$$E_1 = S_0^2e^{2\left(\mu-\frac{\sigma^2}{2}\right)h}P\left(Y - \widetilde{X} < k, Z - \widetilde{X} < k\right),$$

where $(Y - \widetilde{X}, Z - \widetilde{X})$ follows a bivariate normal distribution. $Y - \widetilde{X} \sim \mathcal{N}(-2\sigma h, T)$, $Z - \widetilde{X} \sim \mathcal{N}(-2\sigma h, T)$ and the correlation is $\rho = \frac{h}{T}$. Thus,

$$E_1 = S_0^2 e^{2\mu h} e^{\sigma^2 h} \Phi_2 \left(\gamma, \gamma, \frac{h}{T} \right),$$

where

$$\gamma := \frac{\ln \left(\frac{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)h}}{K} \right) + \left(r + \frac{\sigma^2}{2} \right)(T-h) + 2\sigma^2 h}{\sigma \sqrt{T}}$$

and $\Phi_2(a, b, \rho)$ is the probability that $X < a$ and $Y < b$ when (X, Y) are two standard $\mathcal{N}(0,1)$ r.v. following a bivariate normal distribution with correlation ρ. The expression could be simplified as follows:

$$E_1 = S_0^2 e^{2\mu h} e^{\sigma^2 h} \Phi_2 \left(\widetilde{d_1} + \frac{\sigma h}{\sqrt{T}},\ \widetilde{d_1} + \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T} \right)$$

where $\widetilde{d_1}$ is defined earlier (see expression (17)).

Computation of the second term $E_2 := \mathbb{E}_P \left[S_h \Phi(d_1) \Phi(d_2) \right]$.

One can also write it as

$$E_2 = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)h} \mathbb{E}_P \left[e^{\sigma X} \, \Phi \left(\frac{X + k}{\sqrt{T-h}} \right) \Phi \left(\frac{X + p}{\sqrt{T-h}} \right) | \mathcal{F}_t \right],$$

where $X = W_h$ is $\mathcal{F}_h$-measurable follows $\mathcal{N}(0, h)$ and where k and p are given by

$$\begin{cases} k := \frac{1}{\sigma} \ln \left(\frac{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)h}}{K} \right) + \left(r + \frac{\sigma^2}{2} \right) \frac{(T-h)}{\sigma}, \\ p := \frac{1}{\sigma} \ln \left(\frac{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)h}}{K} \right) + \left(r - \frac{\sigma^2}{2} \right) \frac{(T-h)}{\sigma}. \end{cases}$$

Note that p and k are both $\mathcal{F}_h$ measurable. Let us now define by Y and Z two random variables independent of X and of the σ-field $\mathcal{F}_h$ that both follow $\mathcal{N}(0, T-h)$. E_2 could be written in terms of integrals,

$$E_2 = S_0 e^{\mu h - \frac{\sigma^2}{2} h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\sigma x} \, \mathbb{1}_{y - x < k} \mathbb{1}_{z - x < p} g(x, 0, h) g(y, 0, T) g(z, 0, T) dx dy dz,$$

where $g(x, \mu_x, \sigma_x^2)$ denotes the pdf of $\mathcal{N}(\mu_x, \sigma_x^2)$. Using (11), it could be rewritten as:

$$E_2 = S_0 e^{\mu h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbb{1}_{y-x<k} \mathbb{1}_{z-x<p} g(x, \sigma h, h) g(y, 0, T) g(z, 0, T) dx dy dz.$$

Denote by $\widetilde{X} = X + \sigma h$. The above expression could now be written as:

$$E_2 = S_0^2 e^{\mu h} P\left(Y - \widetilde{X} < k, Z - \widetilde{X} < p\right),$$

where $(Y - \widetilde{X}, Z - \widetilde{X})$ follows a bivariate normal distribution. $Y - \widetilde{X} \sim \mathcal{N}(-\sigma h, T)$, $Z - \widetilde{X} \sim \mathcal{N}(-\sigma h, T)$ and the correlation is $\rho = \frac{h}{T}$. Thus, $E_2 = S_0 e^{\mu h} \Phi_2\left(\frac{k+\sigma h}{\sqrt{T}}, \frac{p+\sigma h}{\sqrt{T}}, \frac{h}{T}\right)$. The expression could be simplified as

$$E_2 = S_0 e^{\mu h} \Phi_2\left(\widetilde{d}_1,\ \widetilde{d}_2 + \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right)$$

where $\widetilde{d}_2$ was defined earlier by (17).

Computation of the third term E_3: Similarly, one obtains

$$E_3 = \Phi_2\left(\widetilde{d}_2,\ \widetilde{d}_2,\ \frac{h}{T}\right).$$

Second moment of the Put Option

Similarly, one can prove that

$$\mathbb{E}_P\left[\mathcal{P}_h^2\right] \quad = \quad F_1 - 2Ke^{-r(T-h)} F_2 + K^2 e^{-2r(T-h)} F_3$$

where the 3 expectations F_i are defined as follows:

$$F_1 := \mathbb{E}_P\left[S_h^2 \Phi(-d_1)^2\right],\ F_2 := \mathbb{E}_P\left[S_h \Phi(-d_1)\Phi(-d_2)\right],\ F_3 := \mathbb{E}_P\left[\Phi(-d_2)^2\right]$$

Omitting the details, one obtains

$$\begin{cases} F_1 &= S_0^2 e^{2\mu h} e^{\sigma^2 h} \Phi_2\left(-\widetilde{d}_1 - \frac{\sigma h}{\sqrt{T}},\ -\widetilde{d}_1 - \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ F_2 &= S_0 e^{\mu h} \Phi_2\left(-\widetilde{d}_1,\ -\widetilde{d}_2 - \frac{\sigma h}{\sqrt{T}},\ \frac{h}{T}\right) \\ F_3 &= \Phi_2\left(-\widetilde{d}_2,\ -\widetilde{d}_2,\ \frac{h}{T}\right). \end{cases}$$

B Computation of the Covariance terms

$$\begin{aligned}
\mathcal{C}_h &= S_h\,\Phi(d_1(K_c)) - K_c\,e^{-r(T-h)}\Phi(d_2(K_c))\\
\mathcal{P}_h &= K_p\,e^{-r(T-h)}\Phi(-d_2(K_p)) - S_h\,\Phi(-d_1(K_p))
\end{aligned}$$

where $d_1(K) = \frac{\ln\left(\frac{S_h}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-h)}{\sigma\sqrt{T-h}}$, $\quad d_2(K) = d_1(K) - \sigma\,\sqrt{T-h}$.

$$\begin{cases}
\begin{aligned}
\mathbb{E}_P\left[\mathcal{C}_h\mathcal{P}_h\right] = \;& K_p e^{-r(T-h)}\mathbb{E}_P\left[S_h\Phi\left(d_1(K_c)\right)\Phi\left(-d_2(K_p)\right)\right]\\
& -K_cK_p e^{-2r(T-h)}\mathbb{E}_P\left[\Phi\left(d_2(K_c)\right)\Phi\left(-d_2(K_p)\right)\right]\\
& -\mathbb{E}_P\left[S_h^2\Phi\left(d_1(K_c)\right)\Phi\left(-d_1(K_p)\right)\right]\\
& +K_c e^{-r(T-h)}\mathbb{E}_P\left[S_h\Phi\left(d_2(K_c)\right)\Phi\left(-d_1(K_p)\right)\right]\\[2ex]
\mathbb{E}_P\left[\mathcal{C}_hS_h\right] = \;& \mathbb{E}_P\left[S_h^2\,\Phi\left(d_1(K_c)\right)\right] - K_c\,e^{-r(T-h)}\mathbb{E}_P\left[S_h\Phi\left(d_2(K_c)\right)\right]\\[2ex]
\mathbb{E}_P\left[\mathcal{P}_hS_h\right] = \;& K_p\,e^{-r(T-h)}\mathbb{E}_P\left[S_h\Phi\left(-d_2(K_p)\right)\right] - \mathbb{E}_P\left[S_h^2\,\Phi\left(-d_1(K_p)\right)\right]
\end{aligned}
\end{cases}$$

The formulas for these expectations are given as follows,

$$\begin{aligned}
\mathbb{E}_P\left[S_h\Phi(d_1)\Phi(-d_2)\right] &= S_0e^{\mu h}\Phi_2\left(\tilde{d}_1(K_c),\ -\tilde{d}_2(K_p)-\frac{\sigma h}{\sqrt{T}},\ -\frac{h}{T}\right)\\
\mathbb{E}_P\left[\Phi(d_2)\Phi(-d_2)\right] &= \Phi_2\left(\tilde{d}_2(K_c), -\tilde{d}_2(K_p),\ -\frac{h}{T}\right)\\
\mathbb{E}_P\left[S_h^2\Phi(d_1)\Phi(-d_1)\right] &= S_0^2e^{2\mu h}e^{\sigma^2h}\Phi_2\left(\tilde{d}_1(K_c)+\frac{\sigma h}{\sqrt{T}},\ -\tilde{d}_1(K_p)-\frac{\sigma h}{\sqrt{T}},\ -\frac{h}{T}\right)\\
\mathbb{E}_P\left[S_h\Phi(-d_1)\Phi(d_2)\right] &= S_0e^{\mu h}\Phi_2\left(-\tilde{d}_1(K_p),\ \tilde{d}_2(K_c)+\frac{\sigma h}{\sqrt{T}},\ \frac{-h}{T}\right)\\
\mathbb{E}_P\left[S_h^2\,\Phi(d_1)\right] &= S_0^2e^{2\mu h}e^{\sigma^2h}\Phi\left(\tilde{d}_1(K_c)+\frac{\sigma h}{\sqrt{T}}\right)\\
\mathbb{E}_P\left[S_h\Phi(d_2)\right] &= S_0e^{\mu h}\Phi\left(\widetilde{d}_2(K_c)+\frac{\sigma h}{\sqrt{T}}\right)\\
\mathbb{E}_P\left[S_h\Phi(-d_2)\right] &= S_0e^{\mu h}\Phi\left(-\widetilde{d}_2(K_p)-\frac{\sigma h}{\sqrt{T}}\right)\\
\mathbb{E}_P\left[S_h^2\,\Phi(-d_1)\right] &= S_0^2e^{2\mu h}e^{\sigma^2h}\Phi\left(-\tilde{d}_1(K_p)-\frac{\sigma h}{\sqrt{T}}\right)
\end{aligned}$$

where

$$\widetilde{d}_1(K) = \frac{\ln\left(\frac{S_0e^{\mu h}}{Ke^{rh}}\right)+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad \widetilde{d}_2(K) = \widetilde{d}_1 - \sigma\sqrt{T}$$

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