Beauty Contests, Risk Shifting, and Bubbles *

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Abstract

In a beauty contest model, the current stock price depends on investors' beliefs of intermediary prices and the final payoff. We show that investors who perceive the lowest risk in one period tend to bear more of the financial risk in that period. As investors who perceive the lowest risk can vary across different periods, the overall perception of risk is reduced in an economy with dynamic trading. The reduction of risk premium causes the stock price to be higher than its fundamental value, resulting in a bubble. If investors with lower risk perceptions of the intermediary prices are more optimistic about the intermediary prices, then the bubble is further enlarged. On the other hand, if investors with lower risk perceptions of the intermediary prices are more pessimistic about the intermediary prices, then a bubble can be reduced or even become negative. Reduction of perceived overall risk due to dynamic trading also increases market liquidity of the stock. Contrary to the conventional wisdom, we show that stock trading volume is also affected by the differences of opinion in future trading periods, due to the demands to hedge future trading opportunities. We extend the setting to incorporate multiple stocks and show that dynamic trading and heterogeneous beliefs about risk lead to a lower risk premium on the market portfolio.

1 Introduction

In a well known metaphor, when investors have heterogeneous beliefs, Keynes (1936) views the stock market akin to a beauty contest in which the contestants have to pick the one that others will pick as the prettiest. Implicit in this argument is that investors care about not only the dividend flows received by holding the stock but also the gains from trading with other investors at intermediary prices. Therefore, investors need to forecast the crowd's forecasts to formulate their own trading strategies.

From the point of valuation, the Keynes beauty contest intuition can affect asset prices through two channels: the cash flow effect and the discount rate effect. In terms of the cash flow effect, investors can differ in their conditional expectations of the future cash flows of a stock. When investors believe that pessimism in the crowd will prevail in the market in the future, they will sell the stock at a very low price today, resulting in a negative bubble. When investors believe that optimism in the crowd will dominate in the future, they will buy the stock at a very high price, resulting in a (positive) bubble. In terms of the discount rate effect, investors can differ in their perception of conditional risk across time. At any time period, investors with lower perceptions end up bearing more risk. As a result, the perceived risk in the market reduces when an investor can trade dynamically and believes that he can shift risk to others at times when his perception of the risk is higher than that of the crowd. It follows that the risk premium or the discount rate required in the economy reduces.

This paper formalizes Keynes' intuition and analyzes the effects of heterogenous beliefs on stock prices. We show that even if all investors agree on the expectation of a stock payoff, as long as they disagree on the risk of the stock payoff, a bubble can still arise due to a lower risk premium in the stock price. In a multiple stock setting, we further show that the heterogeneous beliefs about individual stocks lead to a lower risk premium on the market portfolio. More generally, both bubbles and negative bubbles can occur depending on the size and sign of the cash flow effect.

Specifically, we analyze the effects of future public information disclosure on stock prices and market liquidity in an economy with divergent interpretation of public information. We consider a market with a continuum of risk-averse investors with CARA utility of the same risk aversion. There are $T+1$ dates with trading at time $0, 1, \cdots, T-1$ and consumption at time T. There is one risky stock and one riskfree asset available for trading. Investors have the same beliefs about the stock payoff but disagree on how to interpret public information to be released in future trading sessions. Investors may disagree about both the mean of the public information and the variance-covariance matrix of the public information and the stock payoff. As a result, investors have different beliefs about the expectations and the risks of the intermediary prices. We analyze how dynamic trading, due to the disclosure of future public signals, affects the stock price and the market liquidity at time 0.

We consider three scenarios of differential interpretation of public signals. In the first scenario, investors agree about the the mean of the public information but disagree about the variance-covariance matrix between the stock payoff and the public information. As a result, investors have the same expectation of intermediary prices but different beliefs on the risks of intermediary prices. In the second scenario, investors disagree about the mean of the public information but agree about the variance-covariance matrix of the stock payoff and the public signals. In this case, investors have the same beliefs about the risks but different expectations of the intermediary prices. Lastly, investors disagree about both the expectation of the public information and the variance-covariance matrix between the public information and the stock payoff. As a result, investors have different beliefs about both the expectation and the risk of intermediary prices.

In the first scenario, investors have the same means about the stock payoff and the public information. Consequently, investors have the same expectation about the intermediary prices but different beliefs on the intermediary price risk.[1] Investors, who perceive the lowest risk in one period, tend to hold more shares of the stock in that period, bearing more of the stock risk. As investors may have different views on risks in different periods, investors who have high price risks in one period may have low price risks in the next. As a result, risk is shifted around to the ones who perceive it less, and the overall risk perception in the stock is reduced. Consequently, the risk premium is lower, leading to a higher stock price.

In the second scenario, investors disagree on the expectation of the public information but agree on the variance-covariance matrix between the public signals and the stock payoff. Hence, investors disagree about the expected intermediary prices but agree on the intermediary price risks. It follows that the current stock price depends on the average expectation of the investors' average expectations. Since investors agree on the variance-covariance matrix of the public signals and the stock payoff, the average expectation of the average conditional expectations equals the average unconditional expectation. As a result, the future trading opportunity has no effect on the current stock price.

In the third scenario, investors disagree about both the expectation of the public signals and the variance-covariance matrix of the public signals with the stock payoff. In this case, the risk reduction effect still holds and the risk premium is smaller. However, the average expectation of the average conditional expectations no longer equals the average unconditional expectation. When investors, who are the most optimistic about the intermediary prices, also perceive the lowest risks of the intermediary prices, they will push the stock price up further, resulting in a larger bubble. On the other hand, when investors, who are most pessimistic about the intermediary prices, perceive the lowest intermediary stock price risk, they will pull the price down, reducing the size of the bubble or even resulting in a

[1] In Harrison and Kreps (1978) in which investors are risk neutral and short-sales constraints are present, some investors have the highest expectation in one period or state, whereas other investors have the highest expectation in another period or state. Hence, the stock price today is determined by the highest price in every future state.

negative bubble.

We further analyze the effects of dynamic trading on market liquidity and trading volume. We show that the reduction of perceived risk increases market liquidity. Since investors will trade in future periods due to differential interpretation of public information, the incentive to hedge creates demands proportional to expected returns in future trading periods. Consequently, trading volume is affected by differences of opinion in the past, current and future trading periods. Extending the model to a multiple stock setting, we further demonstrate that heterogeneous beliefs and dynamic trading reduce the perceived risk of the market portfolio, generating a lower risk premium on the market portfolio.

Our paper is related to the seminal paper of Harrison and Kreps (1978), who show that a bubble can form due to resale options in a market with risk-neutral investors and short-sales constraints.[2] In their model, the stock is held by those investors who are the most optimistic about the stock in every future state. Even if an investor has the highest expectation or valuation about the stock payoffs at time 0, he does not necessarily have the highest valuation for the stock in every future state. As a result, there is a resale option as investors can buy the stock expecting that they can sell it to investors who are more optimistic in some future states with a positive probability. Consequently, the stock price is higher than the buy-and-hold value of all investors, or a bubble arises.

The Harrison-Kreps (1978) model and all of its extensions focus on the expectation part of stock prices. For example, this literature illustrates that under heterogeneous expectations, the price of a stock can be higher than the valuation of the most optimistic investor, because the average expectation of all investors can exceed the highest expectation of individual investors. In addition, this literature is typically confined to a one-stock economy, hence,

[2] In a static model, Miller (1977) argues that short-sales constraints can bias the stock price upward as investors with pessimistic views are sidelined, or only optimistic investors participate in the market. Jarrow (1980) shows that Miller's argument holds in a single risky security world but additional conditions are required in a multiple security economy.

it cannot offer any insight on the impact of heterogeneous beliefs on the valuation of the market portfolio. The assumption of risk neutrality also makes it difficult to address how heterogeneous beliefs affect risk premium and market liquidity.

A large literature following Harrison and Kreps (1978) has further analyzed the effects of heterogeneous beliefs on various behavior of stock prices and trading volume. Scheinkman and Xiong (2003) solve the resale option value in close form and demonstrate that the bubble is only mildly affected by transaction cost, as investors can refrain from trading frequently when transaction cost increases. In an important insight, Allen, Morris, and Shin (2006) demonstrate that under heterogeneous expectations due to private information, the law of iterated expectations may not hold for the average expectations. Consequently, the stock price today, which is determined by the average expectation of future average conditional expectations, can exceed the highest valuation by the most optimistic investor in the market, resulting in bubbles. Cao and Ou-Yang (2009a) show that under certain conditions on investors' expectations of other investors' conditional expectations of future dividends, bubbles can arise without short-sales constraints.

Harris and Raviv (1993) and Kandel and Pearson (1995) have analyzed trading volume in stocks with heterogeneous beliefs. Cao and Ou-Yang (2009b) further consider the effects on options trading and show that differential interpretation of public information has different effects on stocks and options trading volume. Due to the restrictive assumption of the specific forms of heterogeneity in their model, trading volume is not affected by differences of opinion beyond the next period. This is in contrast with our results in which trading volume is affected by differences of opinion in all future periods. In addition, we show that trading volume depends also on the past stock returns, which cannot be derived in any other models under differences of opinion.

In short, this paper represents perhaps the first study on how more risk is shifted to the investors who perceive it less under heterogeneous beliefs. Risk shifting results in a lower risk premium in the market. The higher the divergence of investors' views on the risks of stock payoffs and public signals, the lower the risk premium or the higher the expected return on the stock. Our results provide potential explanations for the empirical findings of Diether, Malloy, and Sherbina (2002) and Goetzmann and Massa (2004). For example, Goetzmann and Massa find that the dispersion of opinion of the investors in a stock, which is proxied by investors' age, profession, or income, is positively related to contemporaneous returns and negatively related to its future returns. They also find evidence that dispersion of opinion aggregates across individual stocks, having a market-wide effect.

The rest of this paper is organized as follows. Section 2 describes the basic model. Section 3 extends the model to differential priors and multiple stocks. Section 4 concludes the paper. The appendix contains technical proofs.

2 Basic Model

We consider a T trading session model, with a time line of $0, 1, \cdots, T$. There is one risk free asset and one risky stock available for trading. It is assumed that the financial market is populated by investors with the population size normalized to one, each indexed by i where $i \in [0, 1]$. At time 0, we assume that each investor is endowed with x^i units of the stock and zero units of the bond. Without loss of generality, the interest rate is taken to be zero. The stock payoff at time T is v. The per capita supply of the stock is a positive number denoted by x.

To obtain closed form solutions, we assume that each investor i has a negative exponential utility function, $-\exp(-\gamma W_{Ti})$, where γ is his risk aversion coefficient and W_{Ti} is his terminal wealth or consumption. We assume that v is normally distributed and that all investors believe that the unconditional mean of v is μ, the unconditional variance of v is Σ_v, and the unconditional precision of v is $\Pi = 1/\Sigma_v$.

Investors first trade in session 0. In session t, $t = 1, 2, \cdots, T-1$, signal y_t is revealed. Let

$$Z_T = (v, y_1, y_2, \cdots, y_{T-1}),$$

investor i's variance-covariance matrix and precision matrix of Z_T are denoted by Σ_{Zi} and Π_{Zi}, respectively. Let

$$P_T = v, \;\; R_t = P_t - P_{t-1} \text{ for } t = 1, 2, \cdots, T,$$

$$R(t, T) = (R_t, R_{t+1}, \cdots, R_T)' \text{ for } t = 0, 1, \cdots, T-1.$$

We characterize the equilibrium prices and demands in the multiple trading session economy as follows.

Theorem 1 *There exists a consensus investor who believes that Z_T follows a multi-variate normal distribution. The consensus investor's expectation, variance covariance matrix, and precision matrix of Z_T are denoted by μ_{Zc}, Σ_{Zc}, and Π_{Zc}, respectively. They are given by*

$$\mu_{Zc} \equiv E_c[Z_T] = \Pi_{Zc}^{-1} \int_i \Pi_{Zi} \mu_{Zi} di,$$

$$\Pi_{Zc} = \int_i \Pi_{Zi} di,$$

$$\Sigma_{Zc} = \Pi_{Zc}^{-1},$$

μ_{Zc}, Σ_{Zc}, and Π_{Zc} denote investor i's variance-covariance matrix and precision matrix of Z_T. Given the beliefs of the consensus investor, there exists an equilibrium in which prices (P_t) and demands (D_{ti}) for the stock at time $t = 0, 1, \cdots, T-1$ are described as:

$$P_t = \mu_{vtc} - \gamma \Sigma_{vtc} x, \tag{1}$$

$$D_{ti} = \frac{1}{\gamma} \sum_{j=t+1}^{T} \Pi_{R_{t+1} R_j ti} \mu_{R_j ti}, \tag{2}$$

$$\mu_{R_j ti} = E_{ti}[R_j], \ j = t, t+1, \cdots, T-1, \tag{3}$$

$$\Pi_{R(t+1,T)ti} = \mathrm{Var}_{ti}^{-1}[R(t+1,T)]. \tag{4}$$

Here μ_{vtc} is the first element of the $1 \times T$ μ column vector, Σ_{vtc} is the 1×1 element in the $T \times T$ Σ matrix, and $\Pi_{R_{t+1}R_j ti}$ is the corresponding element of $\Pi_{R(t+1,T)ti}$.

Note that investor i's terminal wealth at time T is given by

$$W_{Ti} = W_{ti} + \sum_{j=t+1}^{T} D_{(j-1)i} R_{ji}.$$

Such a wealth function can also be obtained in a static economy in which investors can trade directly on $(T-t)$ stocks with a return vector $R(t+1,T)$ and the condition that D_{ji} is measurable with respect to the information set at time j. Notice that the optimal demand, in which D_{ji} is measurable with respect to the information set at time t, is given by

$$(D_{ti}^t, D_{(t+1)i}^t \cdots, D_{(T-1)i}^t)' = \frac{1}{\gamma} \Pi_{R(t+1,T)ti} \mu_{R(t+1,T)ti}, \tag{5}$$

$$\mu_{R(t+1,T)ti} = E_{ti}[R(t+1,T)], \ j = t, t+1, \cdots, T-1. \tag{6}$$

Interestingly, the optimal demand obtained in the static economy D_{ti}^t coincides with the optimal demand in the dynamic economy D_{ti}.

Although markets are incomplete, we are able to construct a consensus investor. The equilibrium stock price is determined as if all investors share the same beliefs as the consensus investor. It follows that the effects of dynamic trading boils down to how the beliefs of the consensus investor are affected. As the normal distribution is summarized by the expectation and variance-covariance matrix, we have the following results.

Proposition 1 *The effect of dynamic trading on stock prices can be decomposed into two components: the expectation effect and the risk effect.*

To analyze the effects of heterogeneous expectations and covariances on the stock price, we consider three cases: (i) investors disagree about the covariances of the signals with the stock payoff; (ii) investors disagree about the expectations of the public signals only; and (iii) investors disagree about both the expectations of the signals and the the covariances of the signals with the stock payoff.

2.1 Heterogeneous Variance-Covariance Matrices

Proposition 2 *When investors disagree about the covariances of the public signals with the stock payoff but agree on the expectations of the public signals, the consensus investor's variance of the stock reduces with the number of trading sessions but his expectation of the stock will not be affected. As a result, the risk premium decreases and the stock price increases as more trading sessions are added. Moreover, market liquidity increases with the number of trading sessions.*

To understand the intuition, consider the following example in which there are two trading periods, 0 and 1, and consumption and liquidation occur at period 2. There are two types of investors of equal proportion. Suppose that

$$v = \mu + \beta(y - \mu_y) + \epsilon,$$

where signal y arrives at time 1, μ is the unconditional expectation of v, μ_y is the mean of y, and ϵ is a noise term that follows a normal distribution. Investors believe that y and ϵ are orthogonal and disagree about the variances of y and ϵ but agree about the mean of y. Type I and II investors believe that the variances of y and ϵ are $(\sigma^2_{yI}, \sigma^2_{\epsilon I})$ and $(\sigma^2_{yII}, \sigma^2_{\epsilon II})$, respectively.

For $i = I, II$, at period one, investor i's demand is given by

$$D_{1i} = \frac{E[v|y] - P_1}{\gamma Var[v|y]} = \frac{\mu + \beta(y - \mu_y) - P_1}{\gamma \sigma^2_{\epsilon i}}.$$

For markets to clear, we must have

$$x = \frac{\mu + \beta(y - \mu_y) - P_1}{2\gamma\sigma^2_{\epsilon I}} + \frac{\mu + \beta(y - \mu_y) - P_1}{2\gamma\sigma^2_{\epsilon II}}.$$

We thus have

$$P_1 = \mu + \beta(y - \mu_y) - \gamma\sigma^2_{\epsilon c}x,$$

where

$$\sigma^2_{\epsilon c} = \frac{2\sigma^2_{\epsilon I}\sigma^2_{\epsilon II}}{\sigma^2_{\epsilon I} + \sigma^2_{\epsilon II}}.$$

is the harmonic average of the variance of ϵ among investors.

Moreover, as investors agree on the conditional expectation of v, the capitals gains from trading, $v - P_1 = \epsilon + \gamma\sigma^2_{\epsilon c}x$, are independent of y. As a result, there is no additional hedging demand at time 0, and the optimal demand at time 0 is simply the myopic demand. Let σ_{P1i} denote investor i's variance of P_1 at time 0. We then have $\sigma^2_{P1i} = \beta^2\sigma^2_{yi}$. Investor i's demand for stock at time zero is then given by

$$D_{0i} == \frac{E[P_1 - P_0]}{\gamma\sigma^2_{P1i}} = \frac{\mu - \gamma\sigma^2_{\epsilon c} - P_0}{\gamma\sigma^2_{P1i}}.$$

Market clearing implies that

$$x = \frac{D_{0I} + D_{0II}}{2} = \frac{\mu - \gamma\sigma^2_{\epsilon c} - P_0}{2\gamma\sigma^2_{P1I}} + \frac{\mu - \gamma\sigma^2_{\epsilon c} - P_0}{2\gamma\sigma^2_{P1II}},$$

$$P_0 = \mu - \gamma\sigma^2_{\epsilon c} - \gamma\beta^2\sigma^2_{yc} = \mu - \gamma\sigma^2_{vc},$$

where

$$\sigma^2_{yc} = \frac{2\sigma^2_{yI}\sigma^2_{yII}}{\sigma^2_{yI} + \sigma^2_{yII}}, \quad \sigma^2_{vc} = \sigma^2_{\epsilon c} + \beta^2\sigma^2_{yc}.$$

Notice that the harmonic average is always smaller than the arithmetic average, we thus have

$$\sigma^2_{vc} = \sigma^2_{\epsilon c} + \beta^2\sigma^2_{yc} < \frac{\sigma^2_{\epsilon I} + \sigma^2_{\epsilon II}}{2} + \beta^2\frac{\sigma^2_{yI} + \sigma^2_{yII}}{2} = \frac{\sigma^2_{\epsilon I} + \beta^2\sigma^2_{yI}}{2} + \frac{\sigma^2_{\epsilon II} + \beta^2\sigma^2_{yII}}{2} = \sigma^2_v.$$

Suppose that $\sigma^2_{yI} > \sigma^2_{yII}$, we then have $\sigma^2_{\epsilon I} = \sigma^2_v - \beta^2\sigma^2_{yI} < \sigma^2_v - \beta^2\sigma^2_{yII} = \sigma^2_{\epsilon II}$. In this case, type II investors have lower intermediary price risk in period 0, whereas type I investors have lower intermediary price risk in period 1. Consequently, investors of type I hold more shares of the stock in period 1 while type II hold more shares of the stock in period 0. Both types of investors feel that they can shift the stock risk to the other type in periods in which they perceive higher risk. In other words, they feel that they can diversify risks across time periods. As a result, the risk bearing capacity in the whole dynamic economy is increased. Hence, the risk perceived by the consensus investor is reduced, and the risk premium decreases.

In general, investors may perceive different β's in the regression equation of v on y. Hence, investors need to hedge their trading risk in future periods, so that risk cannot be clearly separated from period to period. Nevertheless, the basic intuition still carry through, that is, investors believe that they can shift the risk to others through dynamic trading, and thus the risk premium goes down.

2.2 Heterogeneous Expectations

Next, we consider the case in which investors disagree on the expectations of the public signals but agree on the variance-covariance matrices. We arrive at the following results:

Proposition 3 *When investors disagree only on the expectations of public signals, the consensus investor's expectation and variance-covariance matrix of the stock payoff is not affected. Consequently, the risk premium, the stock price, and the market liquidity of the stock is not affected by the introduction of future trading sessions.*

To understand the intuition, we use a two-period example. There are two types of investors of equal proportion. They have homogeneous beliefs about the variance-covariance matrix of y and v but heterogeneous expectations of y. Without loss of generality, we can write

$$v = \mu + \beta(y - \mu_{yi}) + \epsilon, \quad i = I, II.$$

For $i = I, II$, at period one, investor i's demand is given by

$$D_{1i} = \frac{\mu + \beta(y - \mu_{yi}) - P_1}{\gamma\sigma_\epsilon^2}.$$

For markets to clear, we must have

$$x = \frac{\mu + \beta(y - \mu_{yI}) - P_1}{2\gamma\sigma_\epsilon^2} + \frac{\mu + \beta(y - \mu_{yII}) - P_1}{2\gamma\sigma_\epsilon^2}.$$

We thus have

$$P_1 = \mu + \beta\left(y - \frac{\mu_{yI} + \mu_{yII}}{2}\right) - \gamma\sigma_\epsilon^2 x.$$

At time 1 the stock price is the average conditional expectation across investors minus a risk premium. The risk premium is the risk aversion coefficient times the harmonic average of investors' conditional precision and the supply of the stock. Because investors have the same conditional variance, the harmonic average reduces to each investors' conditional variance. Moreover, the capital gains from trading in the stock, $v - P_1$, is independent of y. Hence, there is no additional hedging demand at time 0. As a result, the optimal demand at time 0 is simply the myopic demand given by

$$D_{0i} = \frac{\mu + \beta\left(\mu_{yi} - \frac{\mu_{yI}+\mu_{yII}}{2}\right) - \gamma\sigma_\epsilon^2 x - P_0}{\gamma\sigma_{P1}^2}$$

Market clearing implies that

$$x = \frac{D_{0I} + D_{0II}}{2} = \frac{\mu - \gamma\sigma_\epsilon^2 x - P_1}{\gamma\sigma_{P1}^2},$$

$$P_0 = \mu - \gamma\sigma_v^2 x.$$

Notice that when investors have the same variance-covariance matrix, the average of the average conditional expectations equals the average unconditional expectation. Because investors have the same variance-covariance matrix of y and v, the risk premium term is not affected. Consequently, the stock price at time 0 is not affected by dynamic trading.

2.3 Heterogeneous Expectations and Variance-Covariance Matrices

Proposition 4 *When investors disagree on both the mean and the covariance of public information with the stock payoff, the stock price can be either higher or lower than what would obtain in the static setting. That is, either a bubble or a negative bubble can arise.*

As before, we assume that there are two types of investors of equal proportion. Suppose that

$$v = \mu + \beta(y - \mu_y) + \epsilon.$$

Investors believe that y and ϵ are orthogonal and disagree about the variances of y, ϵ and the mean of y. Type I and II investors believe that the variance of y and ϵ are $(\sigma_{yI}^2, \sigma_{\epsilon I}^2)$ and $(\sigma_{yIi}^2, \sigma_{\epsilon II}^2)$, and that the means of y are μ_{yI} and μ_{yII}, respectively.

For $i = I, II$, at period one, investor i's demand is given by

$$D_{1i} = \frac{\mu + \beta(y - \mu_{yi}) - P_1}{\gamma \sigma_{\epsilon i}^2}.$$

For markets to clear, we must have

$$x = \frac{\mu + \beta(y - \mu_{yI}) - P_1}{2\gamma \sigma_{\epsilon I}^2} + \frac{\mu + \beta(y - \mu_{yII}) - P_1}{2\gamma \sigma_{\epsilon II}^2}.$$

We thus have

$$P_1 = \mu + \beta \left(y - \frac{\sigma_{\epsilon II}^2 \mu_{yI} + \sigma_{\epsilon I}^2 \mu_{yII}}{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2} \right) - \gamma \sigma_{\epsilon c}^2 x,$$

where

$$\sigma_{\epsilon c}^2 = \frac{2\sigma_{\epsilon I}^2 \sigma_{\epsilon II}^2}{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2}$$

is the harmonic average of the variance of ϵ among investors.

Notice that the certainty equivalent gain from trading at time 1, $v - P_1$, is independent of y. As a result, there is no additional hedging demand at time 0, and the optimal demand at time 0 is simply the myopic demand. Let σ_{P1i} denote investor i's variance of P_1 at time 0. We then have $\sigma_{P1i}^2 = \beta^2 \sigma_{yi}^2$. Investor i's demand for stock at time zero is given by

$$D_{0i} = \frac{\mu + \beta\left(\mu_{yi} - \frac{\sigma_{\epsilon II}^2 \mu_{yI} + \sigma_{\epsilon I}^2 \mu_{yII}}{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2}\right) - \gamma\sigma_{\epsilon c}^2 - P_0}{\gamma \sigma_{P1i}^2}.$$

Market clearing leads to

$$\begin{aligned} x &= \frac{D_{0I} + D_{0II}}{2} \\ &= \frac{\mu + \beta\left(\mu_{yI} - \frac{\sigma_{\epsilon II}^2 \mu_{yI} + \sigma_{\epsilon I}^2 \mu_{yII}}{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2}\right) - \gamma\sigma_{\epsilon c}^2 - P_0}{2\gamma \sigma_{P1I}^2} + \frac{\mu + \beta\left(\mu_{yII} - \frac{\sigma_{\epsilon II}^2 \mu_{yI} + \sigma_{\epsilon I}^2 \mu_{yII}}{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2}\right) - \gamma\sigma_{\epsilon c}^2 - P_0}{2\gamma \sigma_{P1II}^2}. \end{aligned}$$

The equilibrium stock price at time 0 is then given by

$$P_0 = \mu_c - \gamma\sigma_{\epsilon c}^2 - \gamma\beta^2\sigma_{yc}^2 = \mu_c - \gamma\sigma_{vc}^2,$$

where

$$\mu_c = \mu + \beta(\mu_{y_I} - \mu_{y_{II}})\left(\frac{1}{1 + \sigma_{y_I}^2/\sigma_{y_{II}}^2} - \frac{1}{1 + \sigma_{\epsilon_1}^2/\sigma_{\epsilon_2}^2}\right),$$

$$\sigma_{yc}^2 = \frac{2\sigma_{yI}^2\sigma_{yII}^2}{\sigma_{yI}^2 + \sigma_{yII}^2}, \quad \sigma_{vc}^2 = \sigma_{\epsilon c}^2 + \beta^2\sigma_{yc}^2.$$

As shown before, the harmonic average is always smaller than the arithmetic average, that is,

$$\sigma_{vc}^2 = \sigma_{\epsilon c}^2 + \beta^2\sigma_{yc}^2 < \frac{\sigma_{\epsilon I}^2 + \sigma_{\epsilon II}^2}{2} + \beta^2\frac{\sigma_{yI}^2 + \sigma_{yII}^2}{2} = \frac{\sigma_{\epsilon I}^2 + \beta^2\sigma_{yI}^2}{2} + \frac{\sigma_{\epsilon II}^2 + \beta^2\sigma_{yII}^2}{2} = \sigma_v^2.$$

On the one hand, the shifting of risk to investors who perceive lower risk results in lower risk premium in the market, which pushes the stock price upward and can cause stock bubbles. On the other hand, the differences in expectations of y can reduce the bubble or even lead to a negative bubble.

Without loss of generality, we assume that $\sigma^2_{yI} > \sigma^2_{yII}$, we then have $\sigma^2_{\epsilon I} = \sigma^2_v - \beta^2\sigma^2_{yI} < \sigma^2_v - \beta^2\sigma^2_{yII} = \sigma^2_{\epsilon II}$. It follows that

$$\frac{1}{1+\sigma^2_{y_1}/\sigma^2_{y_2}} < \frac{1}{2} < \frac{1}{1+\sigma^2_{\epsilon_1}/\sigma^2_{\epsilon_2}}.$$

When $\mu_{yI} < \mu_{yII}$, type II investors not only have lower intermediary price risk but also are more optimistic about the intermediary price. Consequently, we arrive at

$$\mu_c = \mu + \beta(\mu_{y_1} - \mu_{y_2})\left(\frac{1}{1+\sigma^2_{y_1}/\sigma^2_{y_2}} - \frac{1}{1+\sigma^2_{\epsilon_1}/\sigma^2_{\epsilon_2}}\right) > \mu,$$

which enlarges the size of the bubble. Recall that the equilibrium stock price increases with the expectation of the consensus investor, which is the precision weighted average of all investors' expectations. When one group of investors perceives lower risk or perceives to have a signal with a higher precision, his expectation carries a higher weight in the consensus investor's expectation. When one group has both a higher expectation and a signal with a higher precision, the consensus' investor's expectation is higher. This is the reason that the size of a bubble is increased.

Suppose that $\sigma^2_{yI} < \sigma^2_{yII}$ and $\mu_{yI} > \mu_{yII}$, type II investors have lower intermediary price risk and are more pessimistic about the intermediary price. We then obtain

$$\mu_c = \mu + \beta(\mu_{y_1} - \mu_{y_2})\left(\frac{1}{1+\sigma^2_{y_1}/\sigma^2_{y_2}} - \frac{1}{1+\sigma^2_{\epsilon_1}/\sigma^2_{\epsilon_2}}\right) < \mu.$$

Note that the expectation effect can reduce the size of the bubble or even lead to a negative bubble. Whether a bubble or a negative bubble occurs depends on the sign of the expectation effect and how it compares to the risk shifting effect. Similarly, when one group of investors not only expects the stock price at time 1 to be very low but also believes its belief to be of very high precision. As a result, the consensus investor's expectation of the time 1 price is low.

2.4 Market Liquidity

When investors are risk averse, a supply shock of the stock will have a price impact. The inverse of the price impact (measured by the derivative of market price with respect to the the stock supply) is a measure of market liquidity. In this subsection, we analyze the effects of dynamic trading on market liquidity. Let

$$\lambda_t \equiv -\partial P_t / \partial x = \Sigma_{vtc}.$$

Let $1/\lambda_t$ denote the market liquidity at trading session t. The consensus investor's conditional variance reduces as the number of trading sessions increases. We have the following result:

Proposition 5 *Market liquidity increases with the number of future trading sessions.*

As investors have more opportunities to trade in the future, the stock price is less sensitive to the stock supply. When investors have disagreements on the variance-covariance matrix of y_t and v, each investor feels that he can sell the stock to others when the news goes against him. As a result, the perceived risk of buying stocks is reduced and market becomes more liquid.

2.5 Trading Volume

In this subsection, we discuss trading volume in the stock. The optimal demand of investor i can be decomposed into two parts:

$$D_{ti} = \frac{1}{\gamma}\left[\Pi_{R_{t+1}R_{t+1}ti}\mu_{R_{t+1}ti} + \sum_{j=t+2}^{T}\Pi_{R_{t+1}R_jti}\mu_{R_jti}\right]. \tag{7}$$

The first part comes from the myopic demand for the expected returns in the current period, whereas the second part comes from the hedging demand due to opportunities to trade in future periods.

Let T_{ti} denote the trading for investor i in period t, $t = 1, ..., T-1$. We have

$$\begin{aligned} T_{ti} &\equiv D_{ti} - D_{(t-1)i} = \frac{1}{\gamma}\left[\sum_{j=t+1}^{T}\Pi_{R_{t+1}R_jti}\mu_{R_jti} - \sum_{j=t}^{T}\Pi_{R_tR_j(t-1)i}\mu_{R_j(t-1)i}\right] \\ &= \frac{1}{\gamma}\left[\sum_{j=1}^{T}\Pi_{R_{t+1}R_ji}\mu_{R_ji} - \sum_{j=1}^{t}\Pi_{R_{t+1}R_ji}R_j - \sum_{j=1}^{T}\Pi_{R_tR_ji}\mu_{R_ji} + \sum_{j=1}^{t-1}\Pi_{R_tR_ji}R_j\right] \\ &= \frac{1}{\gamma}\left[\sum_{k=1}^{T}(\Pi_{R_{t+1}R_ki} - \Pi_{R_tR_ki})\mu_{R_ki} + \sum_{j=1}^{t-1}(\Pi_{R_{t+1}R_ji} - \Pi_{R_tR_ji})R_j - \Pi_{R_tR_{t+1}i}R_t\right]. \end{aligned} \tag{8}$$

The hedging demand causes the trading volume to depend on differences of expected returns in future periods. To summarize, we have the following result.

Proposition 6 *Differences of opinion of public information in all future periods affect trading in the current period.*

It should be noted that our result is in contrast with those of Cao and Ou-Yang (2009b) in which they show that trading volume does not depend on the differences of opinion in future periods beyond the next period. The key difference is that in Cao and Ou-Yang, they assume that the covariance of y and v is the same across investors and that the shocks in

y are independent across different time periods. In that special case, the coefficient in the expression for the hedging demand $\Pi_{R_{t+1}R_j} = \Pi_{R_t R_j}$ for $j \neq \{t, t+1\}$. It follows that, Cao and Ou-Yang (2009a) obtain

$$\begin{aligned} T_{ti} &= \frac{1}{\gamma}\left[\sum_{k=1}^{T}(\Pi_{R_{t+1}R_k} - \Pi_{R_t R_k})\mu_{R_k i} + \sum_{j=1}^{t-1}(\Pi_{R_{t+1}R_j} - \Pi_{R_t R_j})R_j - \Pi_{R_t R_{t+1}} R_t\right] \\ &= \frac{1}{\gamma}\left[(\Pi_{R_{t+1}R_{t+1}i} - \Pi_{R_{t+1}R_t i})\mu_{R_{t+1}i} + (\Pi_{R_{t+1}R_t i} - \Pi_{R_t R_t i})\mu_{R_t i} - \Pi_{R_t R_{t+1}i} R_t\right], \end{aligned} \quad (9)$$

which is unrelated to differences of opinion on expected returns in future trading periods. Cao and Ou-Yang (2009b) further show that $\Pi_{R_t R_{t+1}i}, \Pi_{R_t R_t i}, \Pi_{R_{t+1}R_{t+1}i}$ are not affected by investors' second moments of returns in future trading periods. Consequently, differences of opinion of the second moments of returns in future periods do not affect trading in the current period.

Our model indicates that investors will trade in response not only to past differences of opinion but also to expected future differences of opinions. As a result, we should observe trading volume to pick up many periods before and after the actual news release period. On the contrary, Cao and Ou-Yang (2009a) predicts that trading volume will pick up only one period before the period of news release and many periods afterwards. We further show that trading volume is affected by the current returns as well as past lagged returns while Kandel and Pearson (1995) and Cao and Ou-Yang (2009a) show that trading volume is only affected by current returns using some restrictive assumptions. These distinctions helps us to determine which model provides a better description of trading volume behavior in the stock market.

3 Extensions

In this section, we consider two extensions of the basic model. The first extension allows for heterogeneous priors on the stock payoff among investors and we come to similar results to those in the basic model. The second extension incorporates multiple stocks.

3.1 Differential Priors

In previous sections, we considered homogeneous priors about the stock payoff. In this section, we extend the model to heterogenous priors and show that the modified version of our results still go through.

At time zero, investor i believes that v is normally distributed with mean of μ_{vi} and variance of Σ_{vi}. The assumptions about their beliefs on $y = (y_1, ..., y_{T-1})$ and the variance-covariance matrix of v and y remain the same as before. The consensus investor's belief is defined in the same way as in Section 2. We have the following result.

Theorem 2 *There exists an equilibrium in which asset prices (P_t) and demands (D_{it}) are described as:*

$$P_t = \mu_{vtc} - \gamma \Sigma_{vtc} x, \tag{10}$$

$$\mu_{vtc} = E_{tc}[v], \quad \Sigma_{vtc} = \mathrm{Var}_{tc}[v], \tag{11}$$

$$\mu_{R_{j+1}ti} = E_{ti}[R_{j+1}], \quad \Pi_{R(t+1,T)ti} = \mathrm{Var}_{ti}^{-1}[R(t+1,T)]. \tag{12}$$

$$D_{ti} = \frac{1}{\gamma}\left[\sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti}\mu_{R_{j+1}ti}\right], \tag{13}$$

where $\Pi_{R_{t+1}R_{j+1}ti}$ *is the corresponding element in* $\Pi_{R_{(t+1},T)ti}$.

We have the following proposition:

Proposition 7 *When investors disagree on how to interpret public information: (i) Dynamic trading always reduces the risk premium in the stock price, which increases the stock price; (ii) Dynamic trading can either increase or decrease the consensus investor's expected stock payoff, which can either increase or decrease the stock price. The stock price will go up (down) when investors optimistic (pessimistic) about the public information also have high precisions about the public information; (iii) Dynamic trading increases market liquidity; (iv) Dynamic trading has no effect on the stock prices when investors agree on the regression coefficient of y_t on v for $t = 1, ..., T-1$.*

Even with differential priors, the same intuition goes through. The mutual insurance effect due to differential beliefs still makes investors feel safer holding the stocks, because they believe that they can benefit from the mistakes made by others in the next round. Consequently, the perceived risk is smaller. The effect on the mean of the consensus investor is the same as before. As long as optimistic (pessimistic) investors also have more precise beliefs, the price tends to be higher (lower). Market liquidity increases as investors believe that they can reduce their exposure to stocks in periods in which they perceive higher risk. As a result, the perceived risk of holding stock is less for all investors which results in higher liquidity. Finally, for there to be no price effect, investors must agree on the regression coefficient of y_t on v for $t = 1, ..., T-1$. In this case y_t can be decomposed into two parts: $y = \beta v + \epsilon$, a hedging component proportional to v and an orthogonal part independent of v. Disagreement on the orthogonal part has no effect on the stock price.

With heterogeneous priors, we can define the fundamental value for investor i as the price that would obtain when all investors agree with investor i about the public information, that is, $P_{0i} = \mu_{vi} - \gamma\Sigma_{vi}x$. We have the following results:

Proposition 8 *The price in the dynamic economy can be higher than the highest fundamental value or lower than the lowest fundamental value. That is, it is possible to have*

$P_0 < \min_i\{P_{0i}\}$ *or* $P_0 > \max_i\{P_{0i}\}$.

To understand the intuition, consider the following numerical example with two types of investors type I and type II of equal proportion, $\mu_{vi} = \mu_v = 1$, for all $i = I, II$. The variances of v and y are 1 and the correlation coefficients of v and y are 0 for type I investors. The variance of v is 1, the variance of y is 3 and the covariance of v and y is 1 for type II investors. The mean of y is μ_{yI} for type I and 0 for type II investors. The supply of the stock is 0.2 and the risk aversion coefficient is 2.

$$\mu_{ZI} = \begin{pmatrix} 1.1 \\ \mu_{yI} \end{pmatrix}, \ \mu_{ZII} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{14}$$

$$\Sigma_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \Sigma_{II} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \tag{15}$$

$$\Pi_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \Pi_{II} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{16}$$

The precision matrix and variance covariance matrix of the consensus investor in the dynamic economy are shown as follows:

$$\Pi_c = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix}, \tag{17}$$

$$\Sigma_c = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{pmatrix}. \tag{18}$$

The expectation of the consensus investor is given by

$$\mu_Z = [\Pi_c]^{-1}[0.5\Pi_I\mu_{ZI} + 0.5\Pi_{II}\mu_{ZII}] = \begin{pmatrix} \frac{7.9}{8} + \frac{1}{8}\mu_{yI} \\ \frac{0.3}{8} + \frac{5}{8}\mu_{yI} \end{pmatrix}. \tag{19}$$

The stock price at time 0 is given by $P_0 = 7.9/8 + 1/8\mu_{yI} - 0.4 * 3/4 = 5.5/8 + 1/8\mu_{yI}$. The stock prices in the static economy without dynamic trading are given by $P_{0I} = 1.1 - 0.4 = 0.7$ and $P_{0II} = 1 - 0.4 = 0.6$, respectively. In the dynamic economy with differences of the covariance among investors, the consensus investor's expectation of the stock payoff now depends on type I investors' expectation of the public information. When μ_{yI} is greater than 0.0125, the stock price at time 0 will be greater than 0.7, resulting in a bubble. When μ_{yI} is less than -0.0875, the stock price is lower than 0.6, resulting in a negative bubble. In other words, the decrease in the consensus investor's expectation offsets the decrease in the risk premium, causing the stock price to be lower. On one hand, heterogeneous beliefs about the covariance of the public information with the stock payoff will always reduce the consensus investor's volatility about the stock, thus increasing the stock price. On the other hand, the consensus investor's expected stock payoff can be either higher or lower than individual investor's expectation. When the consensus investor's expected stock payoff is higher, there is a bubble. When the consensus investor's expected payoff is lower, it depends on whether the volatility or the expected payoff effect dominates. When the latter effect dominates, a negative bubble will result.

3.2 Multiple Stocks

We have analyzed a single stock setting so far. In this subsection, we extend the model to multiple stocks. We assume that there are N stocks represented by an $N \times 1$ random vector V. At time 0, we assume that each investor is endowed with X^i units of the stock and zero units of the bond. The per capita supply of the stocks is a positive vector denoted by X. As before, we assume that investors have negative exponential utility function, $-\exp(-\gamma W_{Ti})$, where γ is the investors' risk aversion coefficient and W_{Ti} is investor i's terminal wealth or consumption at time T. We assume that V is multivariate normally distributed and that investor i believes that the unconditional mean of V is μ_{Vi}, the unconditional variance-covariance matrix of V is Σ_{Vi}, and the unconditional precision matrices of V is Π_{Vi}.

At trading session $t = 1, ..., T-1$, a public signal Y_t of dimension $N \times 1$ arrives. Let the

$N(T-1)\times 1$ vector $Y=(Y_1',...,Y_{T-1}')'$ denote all the public signals. Investor i's variance-covariance matrix about the $NT\times 1$ vector $U=(V',Y_1',...,Y_{T-1}')'$ and Y are Σ_{Ui},Σ_{Yi} and his expectations of Y,U are denoted by μ_{Yi},μ_{Ui}, respectively. Let $P_T=V$, $R_t=P_t-P_{t-1}$, $R(t,T)=(R_t,...,R_T)$. We have the following results:

Theorem 3 *There exists an equilibrium in which demands and stock prices are described as:*

$$P_t=\mu_{Vtc}-\gamma\Sigma_{Vtc}X, \tag{20}$$

$$D_{ti}=\frac{1}{\gamma}\left[\sum_{j=t}^{T-1}\Pi_{R_{t+1}R_{j+1}ti}\mu_{R_{j+1}ti}\right], \tag{21}$$

where $\Pi_{R_{t+1}R_{j+1}ti}$ *is the corresponding element in* $\Pi_{R(t,T)ti}$

$$\mu_{Vtc}=E_{tc}[V],\quad \Sigma_{Vtc}=\mathrm{Var}_{tc}[V], \tag{22}$$

$$\mu_{R_{j+1}ti}=E_{ti}[R_{j+1}],\quad \Pi_{R(t+1,T)ti}=\mathrm{Var}_{ti}[R(t+1,T)], \tag{23}$$

and $\Pi_{R_{t+1}R_{j+1}ti}$ *is the corresponding element of* $\Pi_{R(t+1,T)ti}$*.*

We have the following results regarding the effects of dynamic trading on the risk premium, the stock prices, and the market liquidity.

Proposition 9 *(i) The risk premium on the market portfolio decreases; (ii) When all investors agree on the expectation of the public signals, the price of the market portfolio will increase; (iii) The consensus investor's variance covariance matrix of the stocks will reduce; (iv) Market liquidity will increase; (v) When investors agree on the regression coefficients of the public signal on the stock payoffs, there will be no effect on market liquidity, stock prices and options prices; (vi) When investors agree on the beta of the public signals with the market portfolio, there will be no effect on the price of the market portfolio.*

With multiple stocks, the results are similar to those in the single stock case, with the exception that additional trading sessions reduce the variance-covariance matrix of the consensus investor. In addition, the market portfolio carries a special role in the multiple stock economy, and our results with respect to the market portfolio is similar to that of the stock in a single stock setting. In general, risk premiums on individual stocks can either increase or decrease with additional trading sessions but the risk premium on the market portfolio always reduces. The reduction in the risk premium of the market portfolio implies that the price of the market portfolio will increase when investors agree on the expectation of stock payoffs and public signals. Market liquidity increases with additional trading sessions due to the reduction of the variance-covariance matrix of the consensus investor. Finally, when investors agree on the hedge ratio of the public signals with respect to the the stock payoffs, investors are effectively trading on their differences with respect to the unhedged components in public signals. The introduction of additional trading sessions has no effect on risk premium, stock prices, or market liquidity. Similar conditions exist for there to be no effect on the price of the market portfolio.

4 Conclusion

With the assumption of risk-averse investors, this paper represents perhaps the first analysis of heterogeneous beliefs on risk premiums. We show that the impact of heterogeneous beliefs on the stock price can be decomposed into two components: the expectation effect and the risk premium effect. The expectation effect arises when investors have different means about the public signals. When investors relatively more optimistic about the public information also believe that their beliefs are more precise, they tend to push the stock price up to create a bubble. On the other hand, when investors relatively pessimistic about the public information have more precise beliefs, they tend to pull the stock price down and can potentially generate a negative bubble. The risk premium effect arises from the heterogeneous beliefs about the stock risks among investors. Different investors perceive different levels of risk across time and states. When some investors perceive high risks, they believe that other investors may perceive low risk so that they can share risks with one another. As a result, all investors believe that they can achieve better risk sharing under heterogeneous beliefs. In equilibrium, the risk premium decreases. We further find lower risk perception increases market liquidity. Moreover, trading in future periods generates hedging demands in the current period and causes trading volume to respond to differential interpretation of public information released in future trading periods. Extending the setting to multiple stocks, we show that heterogeneous beliefs about risks lead to a lower risk premium on the market portfolio.

Appendix

Proof of Theorem 1: First, we have the following lemma.

Lemma 1 *Let X be an M dimensional random vector and Y be an N dimensional random vector and X, Y are multi-variate normally distributed. Let $\Sigma_{X,Y}$ denote the variance co-variance matrix and $\Pi_{X,Y}$ the precision matrix of $(X', Y')'$. Let $\Pi_{X,Y}(X, X)$ be the $M \times M$ submatrix corresponding to X. Let $\Sigma_{X|Y}$ and $\Pi_{X|Y}$ denote the variance covariance matrix and the precision matrix of X conditional on Y respectively. We have $\Pi_{X|Y} = \Pi_{X,Y}(X, X)$*

Proof: Let Σ_X, Σ_Y denote the variance-covariance matrix of X, Y, ΣXY denote the co-variance matrix of X and Y, Σ_{YX} the transpose of Σ_{XY}. From the block matrix inversion formula, we have

$$\Pi_{X,Y}(X, X) = [\Sigma_X - \Sigma_{XY}\Sigma_Y\Sigma_{YX}]^{-1} = [\Sigma_{X|Y}]^{-1} = \Pi_{X|Y}. \tag{24}$$

Following Lemma (1), we have $\Pi_{R_{t+1}R_{j+1}ti} = \Pi_{R_{t+1}R_{j+1}i}$ for $j \geq t$.

We prove the theorem in the following steps. First we construct the consensus investor according the beliefs given in Theorem 1. We then show that the prices and demands in the last period $T - 1$ constitute a unique equilibrium. Finally, we show that if the prices and demands from $t + 1$ and onwards constitute a dynamic equilibrium, the prices and demands at period t also constitute a unique equilibrium. The theorem thus follows by mathematical induction.

Notice that R_j is the differences of conditional expectations for r and thus is independent across j for investor r. Consequently, we have

$$\Sigma_{vjc} - \Sigma_{v(j+1)c} = \sum_{k=j+1}^{T} \Sigma_{R_{k+1}c} - \sum_{k=j+2}^{T} \Sigma_{R_{k+1}c} = \Sigma_{R_{j+1}c}. \tag{25}$$

Also notice that

$$\mu_{R(t+1,T)ti} - \mu_{R(t+1,T)i} = \text{Cov}_i[R(t+1,T), R(1,t)]\Pi_{R(1,t)i}(R(1,t) - \mu_{R(1,t)i}). \tag{26}$$

Let $\Pi_{R_{t+1}R(1,t)i}$ and $\Pi_{R_{t+1}R(t+1,T)i}$ denote the row vectors that represent the first t elements and the last $T-t$ elements of the $t+1$th row vector in $\Pi_{R_{(1,T)i}}$. Let $O(t)$ denotes the t dimensional zero vector.

$$\Pi_{R_{t+1}R(1,t)i}\text{Var}_i[R(1,t)] + \Pi_{R_{t+1}R(t+1,T)i}\text{Cov}_i[R(t+1,T), R(1,t)] = O(t)' \tag{27}$$

as $\Pi_{R(1,T)i}$ is the inverse of $\Sigma_{R(1,T)i}$. We thus have

$$\begin{aligned}
\Pi_{R_{t+1}R(1,t)i}[\mu_{R(t+1,T)ti} - \mu_{R(t+1,T)i}] &= \Pi_{R_{t+1}R(1,t)i}\text{Cov}_i[R(t+1,T), R(1,t)] \times \\
&\quad \Pi_{R(1,t)i}\left[R(1,t) - \mu_{R(1,t)i}\right] \\
&= -\Pi_{R_{t+1}R(1,t)i}\text{Var}_i[R(1,t)]\Pi_{R(1,t)i}\left[R(1,t) - \mu_{R(1,t)i}\right] \\
&= -\Pi_{R_{t+1}R(1,t)i}\left[R(1,t) - \mu_{R(1,t)i}\right]. \qquad (28)
\end{aligned}$$

It can further be shown that the equilibrium demand proposed in Theorem 1 satisfies the following equation:

$$\begin{aligned}
D_{ti} &= \frac{1}{\gamma}\sum_{j=t}^{T-1}\Pi_{R_{t+1}R_{j+1}ti}\mu_{R_{j+1}ti} \\
&= \frac{1}{\gamma}\sum_{j=t}^{T-1}\Pi_{R_{t+1}R_{j+1}i}\mu_{R_{j+1}i} - \frac{1}{\gamma}\sum_{j=0}^{t-1}\Pi_{R_{t+1}R_{j+1}i}(R_{j+1} - \mu_{R_{j+1}i}) \\
&= \frac{1}{\gamma}\sum_{j=0}^{T-1}\Pi_{R_{t+1}R_{j+1}i}\mu_{R_{j+1}i} - \frac{1}{\gamma}\sum_{j=0}^{t-1}\Pi_{R_{t+1}R_{j+1}i}R_{j+1}. \qquad (29)
\end{aligned}$$

At time T, the utility for investor i has the following form:

$$U_{Ti} = -\exp\left\{-\gamma\left[W_{0i} + \sum_{j=0}^{T-1}D_{ji}R_{j+1}\right]\right\}. \tag{30}$$

At time $T-1$, there is only one period left and the problem reduces to a maximization problem in a static setting. Let

$$\mu_{R_T(T-1)i} \equiv \mathrm{E}_{(T-1)i}[R_T] = E_{(T-1)}\left[v - P_{T-1}\right], \tag{31}$$

$$\Pi_{R_T R_T(T-1)i} \equiv \mathrm{Var}^{-1}_{(T-1)i}[R_T]. \tag{32}$$

We have

$$\begin{aligned}
\mathrm{E}_{T-1}[U_{Ti}] \;\; &\propto \;\; -\int_{R_T} \exp\left[-\gamma \sum_{j=0}^{T-1} D_{ji} R_{j+1}\right. \\
&\quad \left. -\frac{1}{2}(R_T - \mu_{R_T(T-1)i})' \Pi_{R_T R_T i}(R_T - \mu_{R_T(T-1)i})\right] dR_T \\
&\propto \;\; -\int_{R_T} \exp\left[-[\gamma D_{(T-1)i} - \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}]R_T - \frac{1}{2}R_T^2\right] dR_T \\
&\propto \;\; -\exp\left[-\frac{[\gamma D_{(T-1)i} - \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}]^2}{2\Pi_{R_T R_T(T-1)i}}\right].
\end{aligned} \tag{33}$$

From the above expected utility function of investor i, we then arrive at the investor's optimal demands for stock:

$$D_{(T-1)i} = \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}. \tag{34}$$

The market clearing condition, $X = \int_i D_{(T-1)i}di$, yields the equilibrium stock price:

$$P_{T-1} = \frac{\int_i \Pi_{R_T R_T(T-1)i}\mu_{v(T-1)i}di - \gamma x}{\int_i \Pi_{R_T R_T(T-1)i}di} = \mu_{v(T-1)r} - \gamma \Sigma_{v(T-1)r}x. \tag{35}$$

That is, the price is the precision weighted average expectation minus the risk premium. Notice that $1/\int_i \Pi_{R_T R_T(T-1)i}di = 1/\int_i \Pi_{R_T R_T(T-1)r} = \Sigma_{v(T-1)r}$.

Suppose that the demand is optimal for trading sessions larger than t. We show that the equilibrium demand is also optimal at session t. Then by mathematical induction, the equilibrium demand described in the theorem is optimal for all trading sessions. Notice that from equation (52), and conditional on the information at time t, we have the following identity for D_{ki}, $k > t$:

$$\begin{aligned} D_{ki} &= \frac{1}{\gamma}\sum_{j=k}^{T-1}\Pi_{R_{k+1}R_{j+1}ki}\mu_{R_{j+1}ki} \\ &= \frac{1}{\gamma}\sum_{j=k}^{T-1}\Pi_{R_{k+1}R_{j+1}ti}\mu_{R_{j+1}ti} - \frac{1}{\gamma}\sum_{j=t}^{k-1}\Pi_{R_{k+1}R_{j+1}ti}(R_{j+1} - \mu_{R_{j+1}ti}) \\ &= \frac{1}{\gamma}\sum_{j=t}^{T-1}\Pi_{R_{k+1}R_{j+1}ti}\mu_{R_{j+1}ti} - \frac{1}{\gamma}\sum_{j=t}^{k-1}\Pi_{R_{k+1}R_{j+1}ti}R_{j+1}. \qquad (36) \end{aligned}$$

Plugging in the expression for $D_{ji},\ j > t$ and taking the expectation with respect to $R(t+1,T)$ at trading session t, we have

$$
\begin{aligned}
\mathrm{E}_{ti}[U_i] \;\; &\propto \;\; -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{T-1} D_{ji} R_{j+1} \\
&\quad -\frac{1}{2}(R(t+1,T)-\mu_{R(t+1,T)ti})'\Pi_{R(t+1,T)ti}(R(t+1,T)-\mu_{R(t+1,T)ti})\Bigg]\, dR(t+1,T) \\
&= \;\; -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
&\quad -\sum_{j=t+1}^{T-1}[\sum_{l=t}^{T-1}\Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1}\Pi_{R_{j+1}R_{l+1}ti}R_{l+1}]R_{j+1} \\
&\quad -\frac{1}{2}(R(t+1,T)-\mu_{R(t+1,T)ti})'\Pi_{R(t+1,T)ti}(R(t+1,T)-\mu_{R(t+1,T)ti})\Bigg]\, dR(t+1,T) \\
&\propto \;\; -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
&\quad -\sum_{j=t+1}^{T-1}\left[\sum_{l=t}^{T-1}\Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1}\Pi_{R_{j+1}R_{l+1}ti}R_{l+1}\right] R_{j+1} \\
&\quad -\frac{1}{2}R(t+1,T)'\Pi_{R(t+1,T)ti}R(t+1,T) + \mu'_{R(t+1,T)ti}\Pi_{R(t+1,T)ti}R(t+1,T)\Bigg]\, dR(t+1,T) \\
&= \;\; -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
&\quad -\sum_{j=t+1}^{T-1}\sum_{l=t}^{T-1}\Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1}\Pi_{R_{j+1}R_{l+1}ti}R_{l+1}]R_{j+1} \\
&\quad -\frac{1}{2}\sum_{j=t}^{T-1}\Pi_{R_{j+1}R_{j+1}ti}R_{j+1}^2 - \sum_{j=t+1}^{T-1}\sum_{l=t}^{j-1}\Pi_{R_{j+1}R_{l+1}ti}R_{l+1}R_{j+1} \\
&\quad +\sum_{j=t}^{T-1}\sum_{l=t}^{T-1}\Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti}R_{j+1}\Bigg]\, dR(t+1,T)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{R(t+1,T)} \exp\left[-\gamma \sum_{j=0}^{t-1} D_{ji} R_{j+1} - [\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}] R_{t+1} \right. \\
&\quad \left. -\frac{1}{2} \sum_{j=t}^{T-1} \Pi_{R_{j+1}R_{j+1}ti} R_{j+1}^2 \right] dR(t+1,T) \\
&\propto -\int_{R_{t+1}} \exp\left[-[\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}] R_{t+1} - \frac{1}{2} \Pi_{R_{t+1}R_{t+1}ti} R_{t+1}^2 \right] dR_{t+1} \\
&\propto -\exp\left[-\frac{[\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}]^2}{2\Pi_{R_{t+1}R_{t+1}ti}}\right]. \qquad (37)
\end{aligned}
$$

The optimal demand can then be determined as

$$
D_{ti} = \frac{1}{\gamma} \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}.
$$

Q.E.D.

Proof of Proposition 1: It follows immediately from equation (1). Q. E. D.

Proof of Proposition 2:

Let $c(T), c(T+1)$ denote the consensus investor in the economy with T and $T+1$ trading sessions. Thus, in the economy, with $T+1$ trading sessions, the stock payoff is realized in period $T+1$ and investors can still trade at session T when a new signal y_T is released. Let $i(T+1)$ denote investor i in the economy with $T+1$ trading sessions, we have

$$
\mathrm{Var}_{c(T+1)}[U(T)] = [\Pi_{U(T)c(T+1)} - \Pi_{U(T)y_T c(T+1)} \Pi^{-1}_{y_T c(T+1)} \Pi_{y_T U(T) c(T+1)}]^{-1} \qquad (38)
$$

$$
\mathrm{Var}_{c(T)}[U(T)] = \left[\int_i \Pi_{U(T)i(T+1)} - \Pi_{U(T)y_T i(T+1)} \Pi^{-1}_{y_T i(T+1)} \Pi_{y_T U(T) i(T+1) di}\right]^{-1}. \qquad (39)
$$

Notice that

$$
\int_i \Pi_{U(T)i(T+1)} di = \Pi_{U(T)c(T+1)}. \qquad (40)
$$

Let $dw(i)$ be a standard Wiener process and define

$$A \equiv \int_i \Pi_{U(T)y_T i(T+1)} \Pi^{-1/2}_{y_T i(T+1)} dw(i),$$

$$B \equiv \int_i \Pi_{U(T)y_T c(T+1)} \Pi^{-1}_{y_T c(T+1)} \Pi^{1/2}_{y_T i(T+1)} dw(i), \quad C \equiv A - B.$$

We have

$$\begin{aligned} \int_i \Pi_{U(T)y_T i(T+1)} \Pi^{-1}_{y_T i(T+1)} \Pi_{y_T U(T) i(T+1)} di &= \mathrm{Var}[A] = \mathrm{Var}[B] + \mathrm{Var}[C] > \mathrm{Var}[B] \\ &= \Pi_{U(T)y_T c(T+1)} \Pi^{-1}_{y_T c(T+1)} \Pi_{y_T U(T) c(T+1)}. \end{aligned} \quad (41)$$

As a result, we must have

$$\mathrm{Var}_{c(T+1)}[U(T)] > \mathrm{Var}_{c(T)}[U(T) \quad (42)$$

which implies that

$$\mathrm{Var}_{c(T+1)}[v)] > \mathrm{Var}_{c(T)}[v]. \quad (43)$$

Q. E. D.

Proof of Proposition 3: In this case, let $y(T-1) = (y_1, ..., y_{t-1})'$, and let $\epsilon = y(T-1) - \beta v$, where

$$\beta = \Sigma_{y(T-1)v} \Sigma_v^{-1}$$

is the regression coefficient of y on v. Let $\Pi_{\epsilon i}$ denote the precision matrix of investor i on ϵ. Since ϵ and v are orthogonal for all investors, we have

$$\Pi_{vc} = \Pi_v, \quad \Pi_{\epsilon c} = \int_i \Pi_{\epsilon i} di \quad (44)$$

which implies that

$$\Sigma_{yc} = \beta \Sigma_v \beta' + \Pi^{-1}_{\epsilon c} \quad (45)$$

Let μ_ϵ denote investor i's expectation of ϵ, we have

$$\mu_{vc} = \mu_v \, \mu_{\epsilon c} = \Pi_{\epsilon c}^{-1} \int \Pi_{\epsilon i} \mu_{\epsilon i} di \tag{46}$$

and

$$\mu_{yc} = \beta \mu_v + \mu_{\epsilon c}. \tag{47}$$

Since the consensus investor's expectation and variance of v do not change with the number of trading sessions, additional trading sessions have no effect on the risk premium, stock price, options prices, and market liquidity.

Q.E.D.

Proof of Proposition 4: The result follows from Lemma 1 and Proposition 2. Q.E.D.

Proof of Proposition 5: The result follows from Proposition 2. Q.E.D.

Proof of Proposition 6: The result follows immediately equation (8). Q.E.D.

Proofs of Proposition 7: The proofs is similar to that of Proposition 3 and is omitted here. Q.E.D.

Proofs of Theorem 2 , The proof is similar to the proof of Theorem 1 and are omitted here. Q. E. D.

Proof of Proposition 8-9: The proofs are similar to that in Proposition 1-5 and are omitted here. Q.E.D.

Proofs of Theorem 3: We prove the theorem in the following steps. First we construct the consensus investor according the beliefs given in Theorem 1. We then show that the prices and demands in the last period $T-1$ constitute a unique equilibrium. Finally, we show that if the prices and demands from $t+1$ and onwards constitute a dynamic equilibrium, the prices and demands at period t also constitute a unique equilibrium. The theorem thus follows by mathematical induction.

Notice that R_j is the differences of conditional expectations for r and thus is independent across j for investor r. Consequently, we have

$$\Sigma_{vjc} - \Sigma_{v(j+1)c} = \sum_{k=j+1}^{T} \Sigma_{R_{k+1}c} - \sum_{k=j+2}^{T} \Sigma_{R_{k+1}c} = \Sigma_{R_{j+1}c}. \tag{48}$$

Also notice that

$$\mu_{R(t+1,T)ti} - \mu_{R(t+1,T)i} = \text{Cov}_i[R(t+1,T), R(1,t)]\Pi_{R(1,t)i}(R(1,t) - \mu_{R(1,t)i}). \tag{49}$$

Let $\Pi_{R_{t+1}R(1,t)i}$ and $\Pi_{R_{t+1}R(t+1,T)i}$ denote the row vectors that represent the first t elements and the last $T-t$ elements of the $t+1$th row vector in $\Pi_{R_(1,T)i}$. Let $O(t)$ denotes the t dimensional zero vector.

$$\Pi_{R_{t+1}R(1,t)i}\text{Var}_i[R(1,t)] + \Pi_{R_{t+1}R(t+1,T)i}\text{Cov}_i[R(t+1,T), R(1,t)] = O(t)' \tag{50}$$

as $\Pi_{R(1,T)i}$ is the inverse of $\Sigma_{R(1,T)i}$. We thus have

$$\begin{aligned}
\Pi_{R_{t+1}R(1,t)i}[\mu_{R(t+1,T)ti} - \mu_{R(t+1,T)i}] &= \Pi_{R_{t+1}R(1,t)i}\text{Cov}_i[R(t+1,T), R(1,t)] \\
&\quad \Pi_{R(1,t)i}(R(1,t) - \mu_{R(1,t)i}) \\
&= -\Pi_{R_{t+1}R(1,t)i}\text{Var}_i[R(1,t)]\Pi_{R(1,t)i}(R(1,t) - \mu_{R(1,t)i}) \\
&= \Pi_{R_{t+1}R(1,t)i}(R(1,t) - \mu_{R(1,t)i}). \qquad (51)
\end{aligned}$$

It can further be shown that the equilibrium demand proposed in Theorem 1 satisfy the following equation:

$$\begin{aligned}
D_{ti} &= \frac{1}{\gamma}\sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti}\mu_{R_{j+1}ti} \\
&= \frac{1}{\gamma}\sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}i}\mu_{R_{j+1}i} - \frac{1}{\gamma}\sum_{j=0}^{t-1} \Pi_{R_{t+1}R_{j+1}i}(R_{j+1} - \mu_{R_{j+1}i}) \\
&= \frac{1}{\gamma}\sum_{j=0}^{T-1} \Pi_{R_{t+1}R_{j+1}i}\mu_{R_{j+1}i} - \frac{1}{\gamma}\sum_{j=0}^{t-1} \Pi_{R_{t+1}R_{j+1}i}R_{j+1}. \qquad (52)
\end{aligned}$$

At time T, the utility for investor i has the following form:

$$U_{Ti} = -\exp\left\{-\gamma\left[W_{0i} + \sum_{j=0}^{T-1} D_{ji}R_{j+1}\right]\right\}. \tag{53}$$

At time $T-1$, there is only one period left and the problem reduces to a maximization problem in a static setting. Let

$$\mu_{R_T(T-1)i} \equiv \mathrm{E}_{(T-1)i}[R_T], \tag{54}$$

$$\Pi_{R_T R_T(T-1)i} = \mathrm{Var}^{-1}_{(T-1)i}[R_T]. \tag{55}$$

We have

$$\begin{aligned}
\mathrm{E}_{T-1}[U_{Ti}] \;\; &\propto \;\; -\int_{R_T} \exp\left[-\gamma \sum_{j=0}^{T-1} D_{ji}R_{j+1}\right. \\
&\quad \left. -\frac{1}{2}(R_T - \mu_{R_T(T-1)i})'\Pi_{R_T R_T i}(R_T - \mu_{R_T(T-1)i})\right] dR_T \\
&\propto \;\; -\int_{R_T} \exp\left[-[\gamma D_{(T-1)i} - \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}]R_T - \frac{1}{2}R_T^2\right] dR_T \\
&\propto \;\; -\exp\left[-\frac{[\gamma D_{(T-1)i} - \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}]^2}{2\Pi_{R_T R_T(T-1)i}}\right].
\end{aligned} \tag{56}$$

We have the following optimal demands for stock:

$$D_{(T-1)i} = \Pi_{R_T R_T(T-1)i}\mu_{R_T(T-1)i}/\gamma \tag{57}$$

and the equilibrium stock price

$$P_{T-1} = \frac{\int_i \Pi_{R_T R_T(T-1)i}\mu_{v(T-1)i}di - \gamma x}{\int_i \Pi_{R_T R_T(T-1)i}di} = \mu_{v(T-1)r} - \gamma\Sigma_{v(T-1)r}x. \tag{58}$$

that is, the price is the precision weighted average expectation minus the risk premium. Notice that $1/\int_i \Pi_{R_T R_T(T-1)i}di = 1/\int_i \Pi_{R_T R_T(T-1)r} = \Sigma_{v(T-1)r}$.

Suppose that the demand is optimal for trading sessions larger than t. We show that the equilibrium demand is also optimal at session t. Then by mathematical induction, the equilibrium demand described in the theorem is optimal for all trading sessions. Notice that from equation (52), and conditional on the information at time t, we have the following identity for D_{ki}, $k > t$:

$$\begin{aligned} D_{ki} &= \frac{1}{\gamma}\sum_{j=k}^{T-1}\Pi_{R_{k+1}R_{j+1}ki}\mu_{R_{j+1}ki} \\ &= \frac{1}{\gamma}\sum_{j=k}^{T-1}\Pi_{R_{k+1}R_{j+1}ti}\mu_{R_{j+1}ti} - \frac{1}{\gamma}\sum_{j=t}^{k-1}\Pi_{R_{k+1}R_{j+1}ti}(R_{j+1} - \mu_{R_{j+1}ti}) \\ &= \frac{1}{\gamma}\sum_{j=t}^{T-1}\Pi_{R_{k+1}R_{j+1}ti}\mu_{R_{j+1}ti} - \frac{1}{\gamma}\sum_{j=t}^{k-1}\Pi_{R_{k+1}R_{j+1}ti}R_{j+1}. \end{aligned} \tag{59}$$

Plugging in the expression for $D_{ji},\ j > t$ and taking the expectation with respect to $R(t+1,T)$ at trading session t, we have

$$
\begin{aligned}
\mathrm{E}_{ti}[U_i] \ \propto\ & -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{T-1} D_{ji} R_{j+1} \\
& -\frac{1}{2}(R(t+1,T)-\mu_{R(t+1,T)ti})'\Pi_{R(t+1,T)ti}(R(t+1,T)-\mu_{R(t+1,T)ti})\Bigg]\, dR(t+1,T) \\
=\ & -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
& - \sum_{j=t+1}^{T-1}[\sum_{l=t}^{T-1} \Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1} \Pi_{R_{j+1}R_{l+1}ti} R_{l+1}]R_{j+1} \\
& -\frac{1}{2}(R(t+1,T)-\mu_{R(t+1,T)ti})'\Pi_{R(t+1,T)ti}(R(t+1,T)-\mu_{R(t+1,T)ti})\Bigg]\, dR(t+1,T) \\
\propto\ & -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
& - \sum_{j=t+1}^{T-1}[\sum_{l=t}^{T-1} \Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1} \Pi_{R_{j+1}R_{l+1}ti} R_{l+1}]R_{j+1} \\
& -\frac{1}{2}R(t+1,T)'\Pi_{R(t+1,T)ti}R(t+1,T) + \mu'_{R(t+1,T)ti}\Pi_{R(t+1,T)ti}R(t+1,T)\Bigg]\, dR(t+1,T) \\
=\ & -\int_{R(t+1,T)} \exp\Bigg[-\gamma \sum_{j=0}^{t} D_{ji} R_{j+1} \\
& - \sum_{j=t+1}^{T-1}[\sum_{l=t}^{T-1} \Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} - \sum_{l=t}^{j-1} \Pi_{R_{j+1}R_{l+1}ti} R_{l+1}]R_{j+1} \\
& -\frac{1}{2}\sum_{j=t}^{T-1} \Pi_{R_{j+1}R_{j+1}ti} R_{j+1}^2 - \sum_{j=t+1}^{T-1}\sum_{l=t}^{j-1} \Pi_{R_{j+1}R_{l+1}ti} R_{l+1} R_{j+1} \\
& + \sum_{j=t}^{T-1}\sum_{l=t}^{T-1} \Pi_{R_{j+1}R_{l+1}ti}\mu_{R_{l+1}ti} R_{j+1}\Bigg]\, dR(t+1,T)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{R(t+1,T)} \exp\left[-\gamma \sum_{j=0}^{t-1} D_{ji} R_{j+1} - [\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}] R_{t+1} \right. \\
&\quad \left. -\frac{1}{2} \sum_{j=t}^{T-1} \Pi_{R_{j+1}R_{j+1}ti} R_{j+1}^2 \right] dR(t+1,T) \\
&\propto -\int_{R_{t+1}} \exp\left[-[\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}] R_{t+1} \right. \\
&\quad \left. -\frac{1}{2} \Pi_{R_{t+1}R_{t+1}ti} R_{t+1}^2 \right] dR_{t+1} \\
&\propto -\exp\left[-\frac{[\gamma D_{ti} - \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}]^2}{2\Pi_{R_{t+1}R_{t+1}ti}}\right]. \qquad (60)
\end{aligned}
$$

The optimal demand can then be determined as

$$
D_{ti} = \sum_{j=t}^{T-1} \Pi_{R_{t+1}R_{j+1}ti} \mu_{R_{j+1}ti}.
$$

Q.E.D.

Proof of Proposition 11: The proof is similar to that in Proposition 1-5 and is omitted here. Q.E.D.

References

Allen, F., S. Morris, and H. S. Shin, 2004, Beauty contests, bubbles, and iterated expectations in asset markets, *Review of Financial Studies*, 19, 719-752.

Cao, H., and H. Ou-Yang, 2009a, Cross expectations and bubbles, Working paper.

Cao, H., and H. Ou-Yang, 2009b, Differences of opinion of public information and speculative trading in stocks and options, *Review of Financial Studies*, 22, 299-335.

Diether, K., C. Malloy, and A. Scherbina, 2002, Differences of opinion and the cross-section of stock returns, *Journal of Finance*, 57, 2113-2141.

Harris, M., and A. Raviv, 1993, Differences of opinion make a horse race, *Review of Financial Studies*, 6, 473-506.

Harrison, J. M., and D. M. Kreps, 1978, Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics*, 93, 323-336.

Jarrow, R. A., 1980, Heterogeneous expectations, restrictions on short sales, and equilibrium asset prices, *Journal of Finance*, 5, 1980.

Kandel, E., and N. D. Pearson, 1995, Differential interpretation of public signals and trade in speculative markets, *Journal of Political Economy*, 103, 831-872.

Keynes, J. M., 1936, The General Theory of Employment, Interest and Money, published by Harcourt, Brace and Company, and printed in the U.S.A. by the Polygraphic Company of America, New York.

Marshall, J. M., 1974, Private Incentives and Public Information, *American Economic Review*, 64, 373-90.

Miller, E. M., 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance*, 32, 1151-1168.

Ofek, E., and M. Richardson, 2003, Dotcom mania: The rise and fall of internet stock prices, *Journal of Finance*, 58, 1113-1137.

Ross, S. A. (1989), "Institutional markets, financial marketing and financial innovation", Journal of Finance 44(3): 541-556.

Scheinkman, J., and W. Xiong, 2003, Overconfidence, short-sale constraints, and bubbles, *Journal of Political Economy*, 111, 1183-1219.

Shiller, 1981, R. J., Do stock prices move too much to be justified by subsequent changes in dividends? *The American Economic Review*, 71, (3), 421-436.

Shiller, 2002, R. J., From efficient market theory to behavioral finance, Cowles Foundation Discussion Paper No. 1385.

2000, Welch, I., Views of financial economists on the equity premium and other issues, *Journal of Business*, 73, 501-537.