Commodities in Dynamic Asset Allocation: Implications of Mean Reverting Commodity Prices*

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Abstract

This paper studies commodity investment in the context of dynamic asset allocation, with a focus on the implications of the commodity return predictability arising from mean reverting commodity prices. The model of financial markets consists of three asset classes: stocks, bonds, and commodities, which generalizes the benchmark setting of Merton (1969). The risk premium in the commodity market is assumed to be dependent on the mean-reverting spot commodity price, and this assumption is supported by the empirical findings of the paper. I solve, in closed form, the optimal portfolio and consumption strategies. The study suggests that allocation to commodities is needed to optimize the instantaneous risk-return profile (myopic purposes), as well as to hedge the stochastic changes of the investment opportunity set (intertemporal purposes). The welfare cost of excluding the commodity from financial decision making is also solved in closed form. A simple numerical exercise shows that there is substantial market timing in the optimal financial policy, and that excluding the asset class of commodities may incur substantial welfare costs, especially for long-term and less risk-averse investors.

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1 Introduction

Commodities have been emerging as an increasingly important class of assets for institutional and individual investors in recent years. Systematic investigation of commodities as an investable asset class goes back at least some 30 years ago [Greer, 1978, Bodie and Rosansky, 1980]. However, the growth of commodity markets to a major alternative investment vehicle is a more recent development. Around 2007, the size of the global commodities derivatives market is estimated to be about 750 billion US dollars [Till and Eagleeye, 2007]. As the markets have grown, more investors have been attracted to commodities. Increased exposure to commodities has been acquired by institutional investors, with pension funds as a notable example, and to a less extent by individual investors as well (see e.g. Mongars and Marchal-Dombrat [2006] and Doyle et al. [2007]).

In the literature of commodity investment, it remains an open question whether and how to include commodities in mainstream portfolios. Studies on commodity investment have generally been based on the performance of investment in commodity futures. The reason is that investment in commodities is mostly by means of derivative products, especially commodity futures, while spot transactions of commodities play little role in commodity investment. Most existing studies on commodity investment apply the one-period mean-variance optimization framework of Markowitz [1952]. In the static mean-variance framework, the key issues investigated by these studies have been whether investment in commodities futures yields a positive risk premium, how such investment covaries with bonds and stocks, and how it hedges against inflation (see e.g. Erb and Harvey [2006], Gordon and Rouwenhorst [2006], Kat and Oomen [2007] and references therein). In this literature, the most controversial issue has presumably been whether or not commodity investment offers a positive risk premium, and if it does, what drives the risk premium. The absence of an appreciably positive risk premium does not necessarily make mean-variance investors refrain from allocating to an asset class, but it will surely make it less attractive or of little practical relevance in most studies. As such, the ongoing debate over the risk premium of commodity futures investment has left it an open question whether or not commodities are an appealing asset class.

Empirical evidence has documented that risk premia in commodity futures markets are timing-varying and predictable. For example, Bessembinder and Chan [1992] show that prices in commodity futures markets can be forecast on the basis of instrumental variables known to possess forecast power in equity and bond markets. Some studies have found that risk premia of commodity investment vary in different states, like the phase of the business cycle, the stance of monetary policy, market sentiment, and the history of investment returns (see, for example, Jensen et al. [2000, 2002], Wang and Yu [2004], Erb and Harvey [2006], Miffre and Rallis [2007], Nijman and Swinkels [2007], and Vrugt et al. [2007]).

Similarly, time-variation and predictability of asset returns have been well documented in the asset classes of stocks and bonds. What are the implications of time-variation and predictability of asset returns for portfolio choice? For the mainstream asset classes, their implications for portfolio choice have been explored in depth, for example, Kim and Omberg [1996] and Wachter [2002] in the case of stocks. In the case of commodities, however, much less research efforts have been devoted to the implications of time-varying and predictable commodity returns for portfolio decision making. In view of this, this paper presents a study of the asset class of commodities in an intertemporal framework, with an explicit focus on the time-varying and predictable returns in commodity markets. In the literature of commodity investment, the closest to this study is presumably Hoevenaars et al. [2008], who address the optimal portfolio policy in the context where expected return of alternative assets, including commodity returns, are time-varying and predictable. For technical reasons, however, Hoevenaars et al. [2008] consider only constant-proportion portfolio strategies, and hence abstract from market timing that in principle will arise from return predictability. This study aims to further the understanding of commodity investment, especially in exploiting commodity return predictability by market timing.

This paper investigates the asset class of commodities in the dynamic optimization framework that was introduced into finance by Merton [1969]. The so-called Merton's problem has been analyzed and extended in various contexts, reflecting different attributes of people's preferences and of financial markets (see, for example, Chapter 9 of Duffie [2001] for a textbook treatment). The literature of dynamic asset allocation, however, has focused predominantly on such traditional asset classes as stocks and bonds. Owing to the growing importance of commodities, it is pertinent to ask, in this established framework, how investors should optimally make their portfolio and consumption decisions when commodities are available in addition to stocks and bonds.

To this end, I introduce into the classical Black-Scholes economy a commodity market. Consistent with the fact that commodity futures are the major commodity investment vehicle, the commodity market is modeled as a futures market. With the addition of the commodity market, the Black-Scholes economy consisting of a riskless bond and a risky stock is augmented to an economy equipped with three asset classes. This three-asset economy, referred to as the "Bond-Stock-Commodity economy" in the following, enables one to capture the richer investment opportunities stemming from the presence of commodities.

The commodity futures market is characterized by a generalized version of the single-factor model in Schwartz [1997]. Following Schwartz [1997], the non-tradable spot commodity price follows a mean-reverting process. Rather than assuming a constant risk premium in the futures market as in Schwartz [1997], I generalize his model by assuming that the risk premium is dependent on the spot commodity price. This generalization can be justified by three reasons. First, as mentioned above, empirical evidence has shown that risk premia in commodity markets are time-varying, and can be predicted by instrumental variables characteristic of the business cycle. Second, empirical studies, for example Fama and French [1988], have identified a strong business cycle component in the variation of spot commodity prices. It suggests that spot commodity prices might have forecast power for risk premia in commodity futures markets. Last but not least, the estimation results of this extended model presented in this paper has provided strong evidence that the effect of the spot price on the risk premium is significant.

In this simple characterization of the asset class of commodities, the risk premium in the commodity market is predicted by the mean-reverting spot commodity price. As is well known, mean reversion is an important property of commodity prices, and mean reversion of prices has become a prevailing assumption in the literature related to the stochastic behavior of commodity prices, for instance, Gabillon [1995], Schwartz [1997], Geman [2005], to name but a few. Moreover, empirical studies of commodity prices have found evidence of mean reversion to various degrees (for example Bessembinder et al. [1995], Pindyck [2001], and Andersson [2007]).

This study, by relating commodity market returns to spot commodity prices, underscores the implications of the mean-reverting nature of the commodity price for commodity investment and portfolio decisions.

The dynamic framework adopted here makes this study distinct from ones that use a static perspective. In the static one-period paradigm, people are assumed to make a *one-off* investment decision at the beginning of the period in order to maximize their utility over the investment outcome at the end of the period. In comparison, the dynamic framework built on an intertemporal setting allows people to make *intermediate* rebalancing. Undoubtedly the dynamic framework offers a richer structure than the static one does, and arguably it is closer to financial decision-making in practice. It has long been known that unless (i) investors have logarithmic utility, or (ii) the financial market offers a constant investment opportunity set in the sense that both the riskfree rate and the market price of risk are constant, the optimal financial policy derived from the dynamic framework is different from the so-called "myopic" policy based solely on one-period analysis. As will be shown, the time variation and predictability of expected returns in commodity markets, once investigated in the framework of dynamic asset allocation, has profound implications for commodity investment and portfolio decisions, which the static mean-variance analysis is unable to accommodate. Therefore, this study, by virtue of a richer framework, will shed new light on the debate on commodity investment, and enable us to expound on some contentious issues arising from static analysis in light of the findings from a dynamic perspective.

This study contributes to the discussion of commodity investment by taking a novel route to approach the issue. Different from focusing on indices of commodity futures in extant literature, I model commodities into the economy as commodity futures underlying those indices. And this approach may have an advantage in comparison with that based on commodity futures indices. It has been recognized that commodity futures indices embed trading strategies of commodity futures (Gordon and Rouwenhorst [2005], Erb and Harvey [2006]). Owing to differing ways of composition, weighting and rebalancing, different commodity indices imply different trading strategies, and hence may well give divergent pictures of commodity investment returns, even in a common time period. In addition to bringing an element of arbitrariness because of varying ways of index building, the application of indices may blur some important characteristics of commodity investment, like the implications of mean reversion in commodity prices. In contrast, this study directly specifies the underlying commodity futures as such, in the hope of achieving a sharp focus on implications of this property.

I shall consider the optimal financial strategy for an investor in two classical cases. In the first, the investor is concerned with maximizing the expected utility over wealth on some fixed horizon date. The second case I consider is that of an investor who derives utilities over life-time consumption. Of these two cases, the first, terminal wealth case only involves portfolio decisions, and is conceptually easier. The intermediate consumption case, being slightly more complicated, involves both portfolio and consumption decisions. In both cases, the investor is assumed to have constant relative risk aversion, and to be more risk averse than a logarithmic investor.

By the specification of the commodity futures market in this article, the risk premium of commodity investment turns out to follow an Ornstein-Uhlenbeck process. This property allows us to approach the dynamic optimization problem in a route similar to that developed by Kim and Omberg [1996] and Wachter [2002], who address the dynamic optimization problem in the context of predictable equity premia. Thanks to the simple structure of the model, the optimal policy and the utility cost of excluding the commodity are solved in closed form. The optimal policy dictates that allocation to commodities is made both for myopic purposes and for intertemporal purposes, whereas stock allocation is made solely out of myopic considerations. The optimal financial strategy involves timing on the spot commodity price, and thus puts forward a theoretical case for the tactical timing strategies studied in some empirical investigation.

The remainder of the article is organized as follows. I present the basic model of the Bond-Stock-Commodity economy in the next section. Section 3 and 4 are devoted to the optimal strategy in the terminal wealth case and in the intermediate consumption case, respectively. In section 5, I estimate the model of commodity futures, and offer some representative numerical examples and discussion. Section

6 concludes.

2 The economy

In the Bond-Stock-Commodity economy, people can invest in three asset classes: bonds, stocks, and commodities. I opt to characterize the traditional asset classes of bonds and stocks by the standard Black and Scholes [1973] model in order to isolate the effect of the introduction of commodities, although it is possible to follow other formulations which were developed in recent years to reflect stochastic interest rates, and the documented predictability of stock returns.

For the riskless bond, the constant interest rate is denoted by r. The stock price S_t follows a geometric Brownian motion

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dZ_{1,t}, \tag{1}$$

where μ_1 and σ_1 are positive constants, and $Z_{1,t}$ is a standard Wiener process.

Following Schwartz [1997], the spot commodity price M_t is specified by the following single-factor model with mean-reverting property:

$$dM_t = \theta(\mu_2 - \ln M_t)M_t dt + \sigma_2 M_t dZ_{M,t}, \tag{2}$$

where θ, μ_2, and σ_2 are positive constants, and $Z_{M,t}$ is another standard Wiener process jointly normally distributed with $Z_{1,t}$. Defining $m_t = \ln M_t$, we have

$$dm_t = \theta\left(\mu_2 - \frac{\sigma_2^2}{2\theta} - m_t\right) dt + \sigma_2 dZ_{M,t}. \tag{3}$$

As in Schwartz [1997], the spot commodity price is assumed to be non-tradable. Assuming the market price of risk associated with the Wiener process driving the spot commodity price, $Z_{M,t}$, is given by

$$\lambda_{M,t} = \alpha + \beta m_t. \tag{4}$$

This assumption is motivated by two empirical findings: (i) expected returns in commodity markets are time-varying and can be predicted by some instrumental variables characteristic of the business cycle; and (ii) there is a strong business cycle component in the variation of spot commodity prices. Note that this model is reduced to the one-factor model of Schwartz [1997] when $\beta = 0$.

Then under the risk-neutral measure

$$dm_t = \left(\widetilde{\mu}_2 - \widetilde{\theta} m_t\right) dt + \sigma_2 d\widetilde{Z}_{M,t},$$

where

$$\widetilde{\theta} := \theta + \sigma_2\beta, \qquad \widetilde{\mu}_2 := \theta\mu_2 - \sigma_2\alpha - \sigma_2^2/2,$$

and $\widetilde{Z}_{M,t}$ is a standard Wiener process under the risk-neutral measure. From the above equation, The distribution of m_T conditioning on m_t $(t < T)$ under the risk-neutral measure is normal with mean and variance:

$$\begin{aligned} E_t^{\mathbb{Q}}[m_T] &= e^{-\widetilde{\theta}t} m_t + \left[1 - e^{-\widetilde{\theta}(T-t)}\right]\frac{\widetilde{\mu}_2}{\widetilde{\theta}} \\ \mathrm{Var}_t^{\mathbb{Q}}[m_T] &= \frac{\sigma_2^2}{2\widetilde{\theta}}\left[1 - e^{-2\widetilde{\theta}(T-t)}\right] \end{aligned}$$

From the martingale property of futures prices under the risk-neutral measure, it follows that the futures price of the commodity with maturity T at time t is

$$F_t(T) = E_t^{\mathbb{Q}}[M_T] = \exp\left(E_0^{\mathbb{Q}}[m_T] + \tfrac{1}{2}\mathrm{Var}_0^{\mathbb{Q}}[m_T]\right).$$

Then,

$$F_t(T) = \exp\left\{e^{-\widetilde{\theta}(T-t)} m_t + \left[1 - e^{-\widetilde{\theta}(T-t)}\right]\frac{\widetilde{\mu}_2}{\widetilde{\theta}} + \frac{\sigma_2^2}{4\widetilde{\theta}}\left[1 - e^{-2\widetilde{\theta}(T-t)}\right]\right\}. \tag{5}$$

To facilitate the solution of Merton's problem, I characterize the asset class of commodities by the self-financing portfolio $\mathcal{M}_t$:

$$d\mathcal{M}_t = \mathcal{M}_t\left(r + \sigma_2\lambda_{M,t}\right) dt + \mathcal{M}_t\sigma_2 dZ_{M,t}. \tag{6}$$

This portfolio is formed by a portfolio strategy of the riskless bond and the commodity futures as follows. Have a long position in the commodity futures, and the futures holding is constantly rolled over to keep the time-to-maturity of futures contracts constant, say equal to ℓ where ℓ is a positive constant. Moreover, the futures holding is such that the notional value of the futures contract (the number of futures contracts times the futures price) at time t is equal to $\mathcal{M}_t e^{\widetilde{\theta}\ell}$. Because futures contracts have zero value, the entire portfolio value $\mathcal{M}_t$ is invested in the riskless bond. For more detail of the portfolio strategy of $\mathcal{M}_t$, see Appendix A.1.

As such, the specification of the Bond-Stock-Commodity market has been completed. To ease the solution to Merton's problem, however, I reformulate this model by converting the two possibly correlated driving Wiener processes to a *standard* two-dimensional Wiener process (i. e. its two components are independent). Denoting the correlation coefficient between $Z_{1,t}$ and $Z_{M,t}$ by ρ where $|\rho| < 1$, the price dynamics of the two classes of risky assets given by (1) and (6) can be rewritten as

$$\begin{aligned} dS_t &= \mu_1 S_t dt + \sigma_1 S_t \begin{bmatrix} 1 & 0 \end{bmatrix} d\boldsymbol{Z}_t, \\ d\mathcal{M}_t &= \mathcal{M}_t(r + \sigma_2\lambda_{M,t})dt + \mathcal{M}_t\sigma_2 \begin{bmatrix} \rho & \bar{\rho} \end{bmatrix} d\boldsymbol{Z}_t, \end{aligned} \tag{7}$$

where $\bar{\rho} := \sqrt{1-\rho^2}$, and $\boldsymbol{Z}_t := \begin{bmatrix} Z_{1,t} & Z_{2,t} \end{bmatrix}^\top$ is a standard two-dimensional Wiener process.[1] In terms of the standard vector Wiener process, m_t can be written as

$$dm_t = \theta\left(\mu_2 - \frac{\sigma_2^2}{2\theta} - m_t\right) dt + \sigma_2 \begin{bmatrix} \rho & \bar{\rho} \end{bmatrix} d\boldsymbol{Z}_t.$$

Given the specification of the financial market as in (7) and the constant interest rate r, the market price of risk associated with the standard vector Wiener process $\boldsymbol{Z}_t$ is

$$\boldsymbol{\lambda}_t = \begin{bmatrix} \lambda_1 \\ \lambda_{2,t} \end{bmatrix},$$

where

$$\begin{aligned} \lambda_1 &= \frac{\mu_1 - r}{\sigma_1}, \\ \lambda_{2,t} &= \lambda_2(m_t) := \frac{\lambda_{M,t}}{\bar{\rho}} - \frac{\rho}{\bar{\rho}}\lambda_1. \end{aligned} \tag{8}$$

That is to say, the market price of risk with respect to $Z_{1,t}$ is constant, whereas that with respect to $Z_{2,t}$ is stochastic and dependent on the commodity price. As such, the investment opportunity set is stochastic in the Bond-Stock-Commodity economy, and the optimal strategy should be different from the myopic one unless the utility function of the investor is logarithmic. Moreover, being a linear transform of m_t, $\lambda_{2,t}$ also follows an Ornstein-Uhlenbeck process:

$$d\lambda_{2,t} = \theta\left(\bar{\lambda}_2 - \lambda_{2,t}\right) dt + \frac{\beta\sigma_2}{\bar{\rho}} \begin{bmatrix} \rho & \bar{\rho} \end{bmatrix} d\boldsymbol{Z}_t, \tag{9}$$

[1] In this article, boldface notation is used to denote vectors and matrices.

where

$$\bar{\lambda}_2 = \frac{\beta}{\bar{\rho}}\left(\mu_2 - \frac{\sigma_2^2}{2\theta}\right) + \frac{\alpha}{\bar{\rho}} - \frac{\rho}{\bar{\rho}}\lambda_1.$$

For a market to exclude arbitrage, it suffices that the Novikov condition holds (see, for example, Chapter 6 in Duffie [2001]):

$$E\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\lambda}_t^\top \boldsymbol{\lambda}_t dt\right)\right] < \infty.$$

It can be verified that the Novikov condition holds indeed in our model,[2] so the Bond-Stock-Commodity economy is free of arbitrage. Moreover, this economy can be shown to be a complete market (see e.g. Kreps and Pliska [1981]), with a unique state-price density ξ_t given by

$$\frac{d\xi_t}{\xi_t} = -rdt - \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t, \quad \text{and} \quad \xi_0 = 1. \tag{10}$$

Now turn to an investor with initial wealth W_0. Her consumption plan is characterized by an consumption-rate process c_t, and her portfolio plan is a process of portfolio weights in the two risky assets $\boldsymbol{x}_t = \left[\begin{array}{cc} x_{S,t} & x_{M,t} \end{array}\right]^\top$, where $x_{S,t}$ and $x_{M,t}$ denote the portfolio weights in the stock, S_t, and in the investable representative commodity, $\mathcal{M}_t$, respectively. The residual, $1 - x_{S,t} - x_{M,t}$, is allocated to the riskless bond.

From the self-financing property of the consumption-portfolio plan, it follows that the wealth process W_t is given by

$$dW_t = W_t\left[r + \boldsymbol{x}_t^\top \boldsymbol{\sigma}\boldsymbol{\lambda}_t\right]dt - c_t dt + W_t \boldsymbol{x}_t^\top \boldsymbol{\sigma} d\boldsymbol{Z}_t, \tag{11}$$

where

$$\boldsymbol{\sigma} = \left[\begin{array}{cc} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\bar{\rho} \end{array}\right].$$

Note that given $x_{M,t}$, we can calculate the corresponding holding of the underlying commodity futures contracts as follows. Recall that in one unit of the representative commodity $\mathcal{M}_t$, the holding of the futures contract with constant time to maturity ℓ has a notional value of $\mathcal{M}_t e^{\widetilde{\theta}\ell}$. When $x_{M,t}$ weight of total wealth is allocated to $\mathcal{M}_t$, it implies the ratio of the notional value of the underlying future contract to the wealth value W_t is $x_{M,t}e^{\widetilde{\theta}\ell}$.

In the terminal wealth case, the investor solves the following dynamic optimization problem:

$$\begin{aligned} &\sup_{\boldsymbol{x}_t} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ \text{s.t.} \quad dW_t &= W_t\left[r + \boldsymbol{x}_t^\top \boldsymbol{\sigma}\boldsymbol{\lambda}_t\right]dt + W_t \boldsymbol{x}_t^\top \boldsymbol{\sigma} d\boldsymbol{Z}_t, \end{aligned} \tag{12}$$

where γ is the constant rate of relative risk aversion. For reasons that will become clear later, γ is assumed to be larger than one throughout the paper to ensure the existence of a well-behaved solution. This assumption is empirically relevant as it is generally supported by empirical studies of people's risk aversion (see e.g. Friend and Blume [1975], Pindyck [1988], and Szpiro [1986]), and by the literature

[2]As λ_1 is constant and hence satisfies the Novikov condition, we only have to verify that it is also true for the Ornstein-Uhlenbeck $\lambda_{2,t}$. Dokuchaev [2007] has proved that a market price of risk following an Ornstein-Uhlenbeck process satisfies the Novikov condition. Following the same reasoning as in Dokuchaev [2007], we can prove that this condition applies in our model.

on the equity premium puzzle. In the intermediate consumption case, the dynamic optimization is

$$\begin{aligned} &\sup_{\boldsymbol{x}_t, c_t} E\left[\int_0^T e^{-\eta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \\ \text{s.t.}\quad dW_t =& W_t\left[r+\boldsymbol{x}_t^\top \boldsymbol{\sigma}\boldsymbol{\lambda}_t\right]dt - c_t dt + W_t \boldsymbol{x}_t^\top \boldsymbol{\sigma} d\boldsymbol{Z}_t, \\ & W_T \geq 0, \end{aligned} \tag{13}$$

where η denotes the subjective discount rate.

3 Pure portfolio optimization

I start with the terminal wealth case, in which there are only portfolio decisions to make. Using the martingale method, the dynamic optimization problem (12) is equivalent to the following static variational problem [Cox and Huang, 1991]:

$$\begin{aligned} &\sup_{\boldsymbol{x}_t} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ \text{s.t.}\quad & W_0 = E\left[\xi_T W_T\right] \end{aligned} \tag{14}$$

In particular, the budget constraint in (12) is equivalent to the static one in (14) that is formulated in terms of the unique state price density. For a solution to this static optimization problem to exist, it suffices that $E\left[\xi_T^{-1}\right]$ is finite [Cox and Huang, 1991], namely the growth-optimal portfolio has a finite expectation. Under the assumption that this condition applies, the optimal terminal wealth is determined by

$$W_T^* = (k\xi_T)^{-1/\gamma},$$

where k is a Lagrange multiplier determined by substituting the optimal terminal wealth into the static budget constraint.

Following Cox and Huang [1989], I define a new variable

$$N_t = (k\xi_t)^{-1}.$$

By Ito's formula,

$$\frac{dN_t}{N_t} = \left(r+\lambda_1^2+\lambda_{2,t}^2\right)dt + \left[\begin{array}{cc}\lambda_1 & \lambda_{2,t}\end{array}\right] d\boldsymbol{Z}_t. \tag{15}$$

From the definition of N_t and the fact that $\xi_t W_t$ is a martingale, we have

$$W_t = \frac{1}{\xi_t}E_t\left[\xi_T W_T^*\right] = \frac{1}{k\xi_t}E_t\left[k\xi_T (k\xi_T)^{-\frac{1}{\gamma}}\right] = N_t E\left[N_T^{\frac{1}{\gamma}-1}\middle|\lambda_{2,t}, N_t\right],$$

where the last equality follows from the fact that $\lambda_{2,t}$ and N_t together form a strong Markov process, and hence $\lambda_{2,t}$ and N_t are all the investor needs to know to evaluate moments of N_T at time t. Therefore, we can define

$$W_t := F(N_t, \lambda_{2,t}, t; T).$$

3.1 Optimal wealth

To simplify notation, I define the following parameters

$$a_1 = 2\left(\frac{1-\gamma}{\gamma}\beta\sigma_2 - \theta\right), \quad a_2 = \frac{1}{\gamma}\left(\frac{\beta\sigma_2}{\bar{\rho}}\right)^2,$$
$$q = \sqrt{a_1^2 - 4\frac{1-\gamma}{\gamma}a_2}, \qquad \lambda^* = \theta\bar{\lambda}_2 + \frac{1-\gamma}{\gamma}\frac{\rho}{\bar{\rho}}\beta\sigma_2\lambda_1. \tag{16}$$

Then the optimal wealth can be presented as follows.

Lemma 3.1 *For an investor concerned with maximizing the expected utility over wealth at time T as described in (12), the optimal wealth is given by*

$$W_t = F(N_t, \lambda_{2,t}, t; T) = N_t^{\frac{1}{\gamma}} H(\lambda_{2,t}, T-t), \tag{17}$$

where

$$H(\lambda_{2,t}, \tau) = \exp\left\{\frac{1}{\gamma}\left[\frac{1}{2}A_1(\tau)\lambda_{2,t}^2 + A_2(\tau)\lambda_{2,t} + A_3(\tau)\right]\right\}, \tag{18}$$

and

$$A_1(\tau) = \frac{1-\gamma}{\gamma}\frac{2\left(1-e^{-q\tau}\right)}{2q - (q+a_1)\left(1-e^{-q\tau}\right)},$$
$$A_2(\tau) = \frac{1-\gamma}{\gamma}\frac{4\lambda^*\left(1-e^{-q\tau/2}\right)^2}{q\left[2q-(q+a_1)\left(1-e^{-q\tau}\right)\right]},$$
$$A_3(\tau) = \int_0^\tau \left[\frac{a_2}{2}A_2^2(x) + \lambda^* A_2(x) + \frac{\gamma a_2}{2}A_1(x) + (1-\gamma)r + \frac{1-\gamma}{2\gamma}\lambda_1^2\right]dx. \tag{19}$$

Proof Given that $W_t = F(N_t, \lambda_{2,t}, t; T)$ and the stochastic differential equations (9) and (15) for $\lambda_{2,t}$ and N_t, we can write the wealth process in the form of a stochastic differential equation by applying Ito's formula:

$$dW_t = \mu_W dt + \sigma_W d\boldsymbol{Z}_t, \tag{20}$$

where

$$\begin{aligned}\mu_W &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial N}N_t\left(r + \lambda_1^2 + \lambda_{2,t}^2\right) + \frac{\partial F}{\partial \lambda_2}\theta\left(\bar{\lambda}_2 - \lambda_{2,t}\right) + \frac{1}{2}\frac{\partial^2 F}{\partial \lambda_2^2}\left(\frac{\beta\sigma_2}{\bar{\rho}}\right)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial N^2}N_t^2(\lambda_1^2 + \lambda_{2,t}^2) - \frac{\partial^2 F}{\partial \lambda_2 \partial N}N_t\frac{\beta\sigma_2}{\bar{\rho}}(\rho\lambda_1 + \bar{\rho}\lambda_{2,t}), \\ \sigma_W &= \frac{\partial F}{\partial N}N_t[\ \lambda_1 \quad \lambda_{2,t}\ ] + \frac{\partial F}{\partial \lambda_2}\frac{\beta\sigma_2}{\bar{\rho}}[\ \rho \quad \bar{\rho}\ ].\end{aligned}$$

Because W_t is a self-financing wealth process, no arbitrage requires

$$\mu_W - rF = \sigma_W \boldsymbol{\lambda}_t.$$

Writing it explicitly leads to the following partial differential equation (PDE)

$$\begin{aligned}&\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial N}N_t + \left(\theta\bar{\lambda}_2 - \theta\lambda_{2,t} - \frac{\beta\sigma_2}{\bar{\rho}}\rho\lambda_1 - \beta\sigma_2\lambda_{2,t}\right)\frac{\partial F}{\partial \lambda_2} + \frac{1}{2}\left(\frac{\beta\sigma_2}{\bar{\rho}}\right)^2\frac{\partial^2 F}{\partial \lambda_2^2} \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial N^2}N_t^2(\lambda_1^2 + \lambda_{2,t}^2) + \frac{\beta\sigma_2}{\bar{\rho}}\frac{\partial^2 F}{\partial \lambda_2 \partial N}N_t(\rho\lambda_1 + \bar{\rho}\lambda_{2,t}) = rF.\end{aligned} \tag{21}$$

F also satisfies the boundary condition,

$$F(N_T, \lambda_{2,T}, T; T) = W_T^*.$$

It is noteworthy that this PDE bears a close resemblance to the PDE for the optimal wealth process in [Wachter, 2002, Eq. (20)], where the optimal portfolio choice problem is addressed in the context of mean-reverting stock risk premia. The resemblance arises from the fact that the market price of risk in Wachter's model is characterized by an Ornstein-Uhlenbeck process, as is $\lambda_{2,t}$ in the Bond-Stock-Commodity model.

The PDE can be solved by first guessing a general form for the solution. Enlightened by the solution to the PDE in Wachter [2002], I guess the form given by (17) and (18). Substituting them back into (21) yields a quadratic equation for $\lambda_{2,t}$; from the fact that both the constant term and the coefficients on $\lambda_{2,t}^2$ and $\lambda_{2,t}$ must be zero, one obtains a system of three ordinary differential equations:

$$\begin{aligned}
\frac{dA_1}{d\tau}(\tau) &= a_2 A_1^2(\tau) + a_1 A_1(\tau) + \frac{1-\gamma}{\gamma}, \\
\frac{dA_2}{d\tau}(\tau) &= a_2 A_1(\tau) A_2(\tau) + \left(\frac{a_1}{2}\right) A_2(\tau) + \lambda^* A_1(\tau), \\
\frac{dA_3}{d\tau}(\tau) &= \frac{a_2}{2} A_2^2(\tau) + \lambda^* A_2(\tau) + \frac{\gamma a_2}{2} A_1(\tau) + (1-\gamma) r + \frac{1-\gamma}{2\gamma}\lambda_1^2.
\end{aligned} \tag{22}$$

Equations of the same form appear in Kim and Omberg [1996] and Wachter [2002], and the solution method is standard. Following Wachter [2002], I assume that $\gamma > 1$ to ensure the existence of a well-behaved solution. Under this assumption, the solution is given by (19).

For the validity of the optimal solution (17), some technical conditions need to be satisfied [Cox and Huang, 1989]. Appendix A.2 verifies that these conditions hold. Therefore the optimal wealth is given by (17). ∎

3.2 Optimal portfolio plan

Turn to the optimal portfolio plan, a plan that secures the optimal wealth. In the martingale solution, the optimal portfolio plan can be obtained by equating the diffusion terms in the two characterizations of the optimal wealth given by (11) and (20). So the optimal strategy can be summarized as follows.

Theorem 3.2 *For an investor facing the problem (12) in the Bond-Stock-Commodity economy, the optimal portfolio plan* $\boldsymbol{x}_t^* := \begin{bmatrix} x_{S,t}^* & x_{M,t}^* \end{bmatrix}^\top$ *is given by*

$$\begin{aligned}
\boldsymbol{x}_t^* &= \frac{1}{W_t}\left(\boldsymbol{\sigma}^\top\right)^{-1}\sigma_W^\top \\
&= \underbrace{\frac{1}{\gamma}\begin{bmatrix} \frac{\lambda_1}{\sigma_1} - \frac{\rho}{\sigma_1\bar{\rho}}\lambda_{2,t} \\ \frac{1}{\sigma_2\bar{\rho}}\lambda_{2,t} \end{bmatrix}}_{\text{myopic part}} + \underbrace{\frac{1}{\gamma}\begin{bmatrix} 0 \\ \frac{\beta}{\bar{\rho}}\left[A_1(T-t)\lambda_{2,t} + A_2(T-t)\right] \end{bmatrix}}_{\text{intertemporal part}}.
\end{aligned} \tag{23}$$

As is standard in the literature, the optimal strategy in the above presentation is decomposed into two parts: a myopic part, and an intertemporal part. The myopic part, independent of investment horizon, is the allocation that an investor would choose if she ignored changes in the investment opportunity set or if her utility function is logarithmic. The interpretation from the perspective of logarithmic utility can be seen directly by setting γ to one: when γ is one, A_1 and A_2 are zero, and the second part disappears. The intertemporal allocation, the concept of which was first introduced by Merton [1971] and repeated in many subsequent studies, depends on the investment horizon and stems from stochastic variations in the investment opportunity set.

Let us look at the allocation to the two risky assets in more detail. Given the empirical evidence presented in Section 5 that the risk premium in the commodity market is decreasing in the spot commodity price, namely a significantly negative estimate of β in (4), it is assumed that $\beta < 0$ in the following discussion. The *stock allocation*, $x^*_{S,t}$, consists solely of a myopic part. It should not come as a surprise, considering that the stochastic changes in the investment opportunity set in the Bond-Stock-Commodity economy are caused by the variations of the commodity price as shown in (8), and hence the commodity should be in a better position to deal with them. The stock allocation, written as a combination of two terms $\frac{\lambda_1}{\gamma\sigma_1} - \frac{\rho}{\gamma\sigma_1\bar{\rho}}\lambda_{2,t}$, has a natural economic interpretation. The first term, $\frac{\lambda_1}{\gamma\sigma_1}$, is the classical stock allocation in the Black-Scholes economy [Merton, 1969]. The second term, $-\frac{\rho}{\gamma\sigma_1\bar{\rho}}\lambda_{2,t}$, is more interesting for our purposes, as it arises from the introduction of the commodity. Because of the relationship between $\lambda_{2,t}$ and the commodity price as given in (4) and (8), the second term implies that the stock allocation is dependent on the commodity price. And the dependence may take three forms, according to the way the commodity and the stock covary with each other: (i) when the stock price is positively correlated with the commodity price ($\rho > 0$), the stock weight is increasing with the commodity price; (ii) when the stock price is negatively correlated with the commodity price ($\rho < 0$), the stock weight is decreasing with the commodity price; and (iii) when they are independent from each other ($\rho = 0$), the stock weight is immune to the commodity price variation, and constant at $\frac{\lambda_1}{\gamma\sigma_1}$.

Different from the case for the stock, the *commodity allocation* $x^*_{M,t}$ is made both for myopic purposes and for intertemporal purposes. First consider the *myopic demand for the commodity*, $\lambda_{2,t}/\gamma\sigma_2\bar{\rho}$. With $\beta < 0$, it is a decreasing function of the commodity price, and the myopic demand requires to sell the commodity when its price rises, and to buy when its price drops. This property follows from the fact that the instantaneous expected return on the commodity is negatively related to the current commodity price. Whether the investor should be long or short the commodity depends on the sign of $\lambda_{2,t}$. Setting $\lambda_{2,t} = 0$, we can solve the threshold value of the commodity price for the myopic demand

$$\bar{M}^{mpc} = \exp\left(\frac{\rho\lambda_1 - \alpha}{\beta}\right).$$

With $\beta < 0$, we can distinguish three cases: (i) when the commodity price is lower than $\bar{M}^{mpc}$, and then $\lambda_{2,t}$ is positive, the myopic demand is a long position; (ii) when the commodity price is greater than $\bar{M}^{mpc}$, and then $\lambda_{2,t}$ is negative, the investor is short the commodity; and (iii) when the commodity price is equal to $\bar{M}^{mpc}$, and then $\lambda_{2,t}$ is zero, the optimal policy dictates no exposure to the commodity for myopic purposes. These are natural results if recalling that the myopic allocation is concerned only with instantaneous returns of assets.

Thus, the myopic demand both for the stock and for the commodity is dependent on the commodity price. This dependence can be accounted for more intuitively by analogy with mean-variance optimization. As is well known (see, for example, Chapter 13 in Ingersoll [1987]), optimal myopic allocation to risky assets, i.e. the portfolio of risky assets optimally chosen by log investors, can be interpreted as the tangency portfolio in the *instantaneous* standard deviation-expectation graph as illustrated in Figure 1. In the Bond-Stock-Commodity economy, the expectation and variance of the return on the stock is fixed, so the stock is characterized by a fixed point in the figure. However, the expectd return on the commodity is

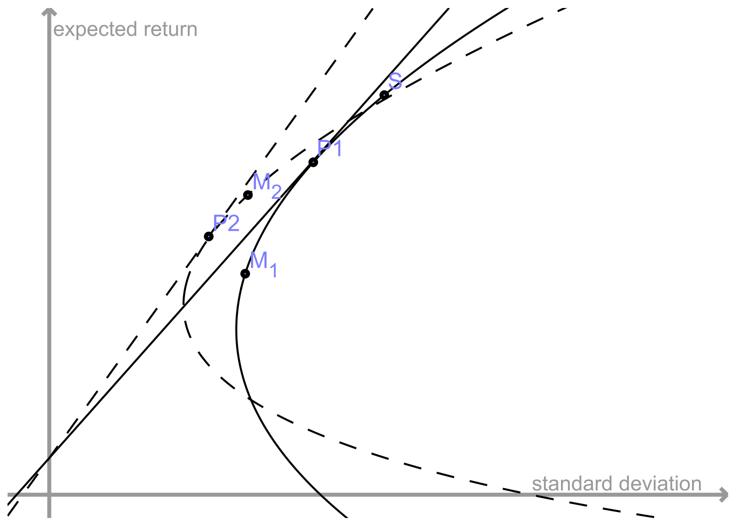

Figure 1: **The dependence of the myopic allocation on the commodity price: an illustration** The figure shows the instantaneous mean-variance optimization for two different commodity price M_1 and M_2. The corresponding tangency portfolios are "P1" and "P2".

conditional on the current commodity price, so the locus of the commodity in the standard deviation-expectation graph is time-varying. Suppose that the spot commodity price is M_1 at a certain time, a myopic investor would find the optimal allocation by looking for the tangency portfolio (labeled as "P1" in the figure), a portfolio based on the fixed locus of the stock and the current locus of the commodity. If the spot commodity price changes to M_2, say, and the commodity changes to a corresponding new locus, then the corresponding new tangency portfolio (labeled as "P2") will be formed according to the "new" commodity. As such, the myopic allocation changes with the variation of the commodity price.

Now turn to the *intertemporal demand for the commodity*. From its expression

$$\frac{\beta}{\gamma\bar{\rho}}\left[A_1(T-t)\lambda_{2,t} + A_2(T-t)\right], \tag{24}$$

it follows that the intertemporal demand changes with the spot commodity price, through the presence of $\lambda_{2,t}$, and with the investment horizon, through the presence of $A_1(T-t)$ and $A_2(T-t)$. First look at the impact of the spot commodity price. Because $\beta < 0$ and $A_1(\tau)$ is negative (see Appendix A.3), the intertemporal allocation is a decreasing function of the commodity price. In other words, as in the myopic allocation to commodity, the intertemporal allocation is to buy the commodity when its price falls, and to sell when its price rises. By definition, the intertemporal allocation is concerned with returns on assets *beyond* the next period (an infinitesimal period in continuous time). For the mean-reverting commodity, a price hike implies that not only the return on the commodity over the next infinitesimal period is getting worse[3], but the returns beyond the next period are deteriorating as well. And it calls for, as a response, lowering the exposure to the commodity beyond what is done is the myopic allocation. Conversely, a price slump implies improved prospects of future returns, and requires an increased exposure in the intertemporal allocation. These observations may help to understand why the intertemporal demand is decreasing in the commodity price.

Consider now the dependence of the intertemporal allocation on the investment horizon. In particular, should a long-term investor allocate more to the commodity

[3]This implication has been captured by the myopic allocation.

than a short-term one? The horizon effect of the intertemporal allocation to the commodity also represents the entire horizon effect of the total risky allocation, as it is the sole element dependent on the horizon. The horizon effect can be characterized by

$$\frac{\beta}{\gamma\bar{\rho}}\left(A_1'(T-t)\lambda_{2,t} + A_2'(T-t)\right), \tag{25}$$

which follows from differentiating (24) with respect to the horizon. As shown in Appendix A.3, $A_1'(\tau)$ is negative, whereas $A_2'(\tau)$ can be positive or negative, depending on the sign of λ^*, where λ^*, as defined in (16), includes the parameters characterizing the financial market and the risk aversion of the investor. Thus, without imposing further constraints on the parameter values, we cannot decide the sign of (25), or the horizon effect. Further discussion of horizon effect will be given in the numerical example in Section 5.

After looking into its two components, we are ready to consider the *total commodity allocation*. First of all, the total commodity allocation is decreasing in the commodity price since both components are decreasing in the price. Another question that one may ask is when the investor should be long or short the commodity as a whole. To answer this question, one can derive the following threshold commodity price for deciding the long/short position of the total commodity allocation

$$\bar{M}^{ttl} = \exp\left[\frac{\rho\lambda_1 - \alpha}{\beta} - \frac{\sigma_2\bar{\rho}A_2(T-t)}{1+\sigma_2\beta A_1(T-t)}\right]. \tag{26}$$

Therefore, when the commodity price is lower, or higher than this threshold price, the investor should be long, or short the commodity, respectively. The threshold $\bar{M}^{ttl}$ depends on the investment horizon, so it is possible that other things being equal, one investor is short the commodity and another is long simply because they have different investment horizons. Unless $\lambda^* = 0$, the threshold in the total commodity allocation, $\bar{M}^{ttl}$, is different from that in the myopic commodity allocation, $\bar{M}^{mpc}$. This highlights the difference between the dynamic framework and the static one: while a myopic investor would be short the commodity, the optimal allocation may be a long position if intertemporal rebalancing is allowed. This difference is attributed to the intertemporal allocation to the commodity, in which the following threshold of deciding a long/short position is used:

$$\bar{M}^{int} = \exp\left[\frac{\rho\lambda_1 - \alpha}{\beta} - \frac{\bar{\rho}A_2(T-t)}{\beta A_1(T-t)}\right]. \tag{27}$$

This threshold is different from $\bar{M}^{mpc}$ unless $\lambda^* = 0$, and it may be less or greater than $\bar{M}^{mpc}$, depending on the sign of λ^*.

3.3 The importance of commodities as an asset class: welfare analysis

For an emerging asset class like commodities, it is natural to ask how important it is to take it into account when making investment decisions. In other words, how costly is it if the new asset class is left out in investment decision-making? To address this question, welfare analysis is applied, as is standard in the literature. In this paper, the importance of incorporating the new asset class, or the utility cost of omitting it, is measured by the percentage extra initial wealth that is necessary to bring the investor to the same expected utility as is obtained by following the optimal strategy. The utility cost can be solved in closed form.

Proposition 3.3 *Suppose that in the Bond-Stock-Commodity economy, an investor is concerned with maximizing the expected utility over wealth at time T as described* *in (12). If the commodity is excluded in portfolio decisions, then L percent of extra initial wealth is needed to achieve the same expected utility level as is obtained by following the optimal strategy (23), and*

$$\frac{L}{100} = \exp\left\{\frac{1}{1-\gamma}\left[\frac{1}{2}A_1(T)\lambda_{2,0}^2 + A_2(T)\lambda_{2,0} + \int_0^T \left(\frac{a_2}{2}A_2^2(x) + \lambda^* A_2(x) + \frac{\gamma a_2}{2}A_1(x)\right) dx\right]\right\} - 1. \quad (28)$$

Proof First, we need to know the expected utility from the optimal strategy, namely the indirect utility function. Cox and Huang [1989] show that the indirect utility function $J(W_t, \lambda_{2,t}, t)$ satisfies the differential equation

$$\frac{\partial J}{\partial W} = \frac{1}{N_t}.$$

From (17), it follows that

$$\frac{1}{N_t} = W_t^{-\gamma} H(\lambda_{2,t}, T-t)^{\gamma}.$$

Then the differential equation becomes

$$\frac{\partial J}{\partial W} = W_t^{-\gamma} H(\lambda_{2,t}, T-t)^{\gamma}.$$

Therefore, the boundary condition $J(W_T, \lambda_{2,T}, T) = W_T^{1-\gamma}/(1-\gamma)$ implies that the indirect utility function is

$$J(W_t, \lambda_{2,t}, t) = \frac{W_t^{1-\gamma}}{1-\gamma} H(\lambda_{2,t}, T-t)^{\gamma}.$$

When the commodity is left out, the Bond-Stock-Commodity economy is reduced to the standard Black-Scholes economy. It is well known that in this specification, the indirect utility is

$$J^{BS}(W_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left[\left(r(1-\gamma) + \frac{1}{2}\frac{1-\gamma}{\gamma}\lambda_1^2\right)(T-t)\right].$$

From the definition of L

$$J^{BS}\left(\frac{100+L}{100}W_0, 0\right) = J(W_0, \lambda_{2,0}, 0),$$

(28) follows immediately. ∎

The exponent in the right hand side of (28) is a quadratic function of the initial commodity price. From the quadratic form, it follows that with either very high or very low commodity prices, the welfare loss is relatively large, whereas with intermediate commodity prices, the loss is relatively small. The effect on the welfare loss of other factors, like the investment horizon and the risk aversion of the investor, will be discussed in the numerical illustrations in Section 5.

4 Optimal portfolio and consumption decisions

Now consider the case where the investor derives utility from intermediate consumption. In this case, apart from deciding what asset mix to hold, the investor needs to decide what fraction of wealth to consume. Thus, assuming utility over consumption allows Merton's problem to be related to people's financial decisions in a way that the previous terminal wealth case does not. On the other hand, the intermediate-consumption case has a close link with the terminal wealth case, in that the the single optimization problem in the former case can be thought of as a series of optimization problems for a continuum of future dates [Wachter, 2002]. In particular, the investor with utility over consumption decides the optimal series of consumption events, and then applies the terminal wealth analysis to each future consumption event. This is analogous to the equivalence between a bond that pays coupon continuously and a continuum of zero-coupon bonds. In the following martingale solution, I shall use this insight and follow a procedure similar to Wachter [2002].

The martingale approach transforms the budget constraint in (13) into a static one,

$$W_0 = E\left[\int_0^T \xi_t c_t dt\right]. \tag{29}$$

The optimal consumption plan follows from the first order condition of the optimization problem in static form,[4]

$$c_t^\star = (\mathcal{K}\xi_t)^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta t}, \tag{30}$$

where $\mathcal{K}$ is a Lagrange multiplier determined by inserting $c_t^\star$ into (29).

The optimal portfolio plan is determined so as to meet the need to finance the consumption plan (30). The wealth at time t, denoted by $\mathcal{W}_t$, is the discounted value of future consumption till time T,

$$\mathcal{W}_t = \frac{1}{\xi_t} E_t\left[\int_t^T \xi_s c_s^\star ds\right].$$

As in the terminal wealth case, define a new variable

$$\mathcal{N}_t = (\mathcal{K}\xi_t)^{-1}.$$

It follows from Ito's formula that

$$\frac{d\mathcal{N}_t}{\mathcal{N}_t} = \left(r + \lambda_1^2 + \lambda_{2,t}^2\right) dt + \begin{bmatrix} \lambda_1 & \lambda_{2,t} \end{bmatrix} d\boldsymbol{Z}_t. \tag{31}$$

That is, $\mathcal{N}_t$ has the same dynamics as N_t, but with a different initial value. From the introduction of $\mathcal{N}_t$, and the strong Markov property of $\begin{bmatrix} \mathcal{N}_t & \lambda_{2,t} \end{bmatrix}^\top$, it follows that

$$\mathcal{W}_t = \mathcal{N}_t E\left[\int_t^T \mathcal{N}_s^{\frac{1}{\gamma}-1} e^{-\frac{1}{\gamma}\eta s} ds \middle| \lambda_{2,t}, \mathcal{N}_t\right].$$

Therefore one can define

$$\mathcal{W}_t := \mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t; T).$$

[4]Here we work under the same technical assumption that $E(\xi_T^{-1})$ is finite as in the terminal wealth case. The other technical conditions for the solution's validity are proved in Appendix A.2.

4.1 Optimal wealth

The optimal wealth assuming interim consumption can be characterized as follows.

Lemma 4.1 *For an investor concerned with maximizing the expected utility over life-time consumption as described in (13), the optimal wealth is given by*

$$\mathcal{W}_t = \mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t; T) = \mathcal{N}_t^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}t} \int_t^T \mathcal{H}(\lambda_{2,t}, s-t) ds, \tag{32}$$

where

$$\mathcal{H}(\lambda_{2,t}, s-t) = \exp\left\{ \frac{1}{\gamma} \left[\tfrac{1}{2}\mathcal{A}_1(s-t)\lambda_{2,t}^2 + \mathcal{A}_2(s-t)\lambda_{2,t} + \mathcal{A}_3(s-t) \right] \right\}, \tag{33}$$

and

$$\begin{aligned} \mathcal{A}_1(\tau) =& A_1(\tau), \\ \mathcal{A}_2(\tau) =& A_2(\tau), \\ \mathcal{A}_3(\tau) =& A_3(\tau) - \eta\tau \end{aligned} \tag{34}$$

Proof Applying Ito's formula to $\mathcal{W}_t = \mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t; T)$, one has

$$d\mathcal{W}_t = \mu_{\mathcal{W}} dt + \sigma_{\mathcal{W}} d\boldsymbol{Z}_t, \tag{35}$$

where

$$\begin{aligned} \mu_{\mathcal{W}} \quad &= \quad \frac{\partial \mathcal{G}}{\partial t} + \frac{\partial \mathcal{G}}{\partial \mathcal{N}} \mathcal{N}_t \left(r + \lambda_1^2 + \lambda_{2,t}^2 \right) + \frac{\partial \mathcal{G}}{\partial \lambda_2} \theta \left(\bar{\lambda}_2 - \lambda_{2,t} \right) + \frac{1}{2} \frac{\partial^2 \mathcal{G}}{\partial \lambda_2^2} \left(\frac{\beta \sigma_2}{\bar{\rho}} \right)^2 \\ &\qquad + \frac{1}{2} \frac{\partial^2 \mathcal{G}}{\partial \mathcal{N}^2} \mathcal{N}_t^2 (\lambda_1^2 + \lambda_{2,t}^2) - \frac{\partial^2 \mathcal{G}}{\partial \lambda_2 \partial \mathcal{N}} \mathcal{N}_t \frac{\beta \sigma_2}{\bar{\rho}} (\rho \lambda_1 + \bar{\rho} \lambda_{2,t}), \\ \sigma_{\mathcal{W}} \quad &= \quad \frac{\partial \mathcal{G}}{\partial \mathcal{N}} \mathcal{N}_t [\ \lambda_1 \quad \lambda_{2,t}\ ] + \frac{\partial \mathcal{G}}{\partial \lambda_2} \frac{\beta \sigma_2}{\bar{\rho}} [\ \rho \quad \bar{\rho}\ ]. \end{aligned}$$

Different from the terminal wealth case, the portfolio process $\mathcal{W}_t$ is not self-financing, since a continuous consumption flow $c_t^\star$ is withdrawn. Thus $\mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t)$ itself does not satisfy the generalized Black-Scholes equation. Instead, in this case no arbitrage requires

$$\mu_{\mathcal{W}} + c_t^\star - r\mathcal{G} = \sigma_{\mathcal{W}} \boldsymbol{\lambda}_t.$$

Writing it explicitly, we have the following PDE for $\mathcal{G}$,

$$\begin{aligned} &\frac{\partial \mathcal{G}}{\partial t} + r \frac{\partial \mathcal{G}}{\partial \mathcal{N}} \mathcal{N}_t + \left(\theta \bar{\lambda}_2 - \theta \lambda_{2,t} - \frac{\beta \sigma_2}{\bar{\rho}} \rho \lambda_1 - \beta \sigma_2 \lambda_{2,t} \right) \frac{\partial \mathcal{G}}{\partial \lambda_2} + \frac{1}{2} \left(\frac{\beta \sigma_2}{\bar{\rho}} \right)^2 \frac{\partial^2 \mathcal{G}}{\partial \lambda_2^2} \\ &\quad + \frac{1}{2} \frac{\partial^2 \mathcal{G}}{\partial \mathcal{N}^2} \mathcal{N}_t^2 (\lambda_1^2 + \lambda_{2,t}^2) + \frac{\beta \sigma_2}{\bar{\rho}} \frac{\partial^2 \mathcal{G}}{\partial \lambda_2 \partial \mathcal{N}} \mathcal{N}_t (\rho \lambda_1 + \bar{\rho} \lambda_{2,t}) \mathcal{N}_t^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma} t} = r\mathcal{G}, \end{aligned} \tag{36}$$

with the boundary condition,

$$\mathcal{G}(\mathcal{N}_T, \lambda_{2,T}, T) = 0.$$

Because a PDE of similar form has been solved by Wachter [2002], I take (32) and (33) as the guessed form of solution here. Substituting them into (36) and matching the coefficients of $\lambda_{2,t}^2$, $\lambda_{2,t}$ and the constant term produces a system of three differential equations very similar to (22). And their solution is (34). ∎

At first glance, it seems hard to understand why the differential equation (36) should have a solution in the integral form as in (32). This guessed solution, however, may follow naturally when utilizing the link between the intermediate consumption analysis and the terminal wealth analysis. Consider a series of auxiliary investors deriving utility from terminal wealth at time $i \in [0, T]$, and each investor is indexed by her fixed horizon date i. Suppose that investor i has initial wealth

$$W_{i,0} = W_0 \frac{E\left[\xi_i^{1-\frac{1}{\gamma}}\right] e^{-\frac{\eta}{\gamma} i}}{E\left[\int_0^T \xi_t^{1-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta t} dt\right]}. \tag{37}$$

Applying the terminal wealth analysis to investor i, we can write her optimal wealth at $t \in [0, i]$ as

$$\mathcal{W}_{i,t} := \mathcal{F}(\mathcal{N}_t, \lambda_{2,t}, t; i) = e^{-\frac{\eta}{\gamma} i} \mathcal{N}_t E\left[\mathcal{N}_i^{\frac{1}{\gamma}-1} \middle| \lambda_{2,t}, \mathcal{N}_t\right]$$

With the introduction of $\mathcal{F}(\mathcal{N}_t, \lambda_{2,t}, t; i)$, the optimal wealth of the investor with utility over consumption $\mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t)$ can be characterized as the sum of the optimal wealth of the auxiliary investors:

$$\mathcal{G}(\mathcal{N}_t, \lambda_{2,t}, t) = \int_t^T \mathcal{F}(\mathcal{N}_t, \lambda_{2,t}, t; s) ds. \tag{38}$$

The terminal wealth analysis can yield a solution of $\mathcal{F}$ that is similar to F given by (17). So the solution (32) follows immediately.

The derivation through a series of auxiliary investors has an interesting economic interpretation. $W_{i,0}$ given in (37) is the value at time zero of the optimal terminal wealth at time i. From (29) and (30), it follows that $W_{i,0}$ is the value at time zero of the optimal consumption event at period i for the investor concerned with intermediate consumption. The fraction at the right hand side of (37) is the ratio of period-i consumption to her life-time consumption in terms of the present value. Therefore, it is correct to think of the investor as holding separate accounts for each future consumption event, distributing her initial wealth into each account according to (37) to achieve the optimal consumption plan, and then investing each account so that the consumption needs are met.

4.2 Optimal portfolio and consumption policy

In the intermediate consumption case, the optimal financial strategy is characterized by the following theorem.

Theorem 4.2 *Suppose that in the Bond-Stock-Commodity economy, an investor seeks to maximize the expected utility over life-time consumption by choosing consumption and investment plans, as formalized in (13). Then the optimal consumption plan can be characterized by the following consumption-wealth ratio,*

$$\frac{c_t^\star}{\mathcal{W}_t} = \left[\int_t^T \mathcal{H}(\lambda_{2,t}, s-t) ds\right]^{-1}. \tag{39}$$

The optimal portfolio plan, denoted by $\boldsymbol{x}_t^\star := \begin{bmatrix} x_{S,t}^\star & x_{M,t}^\star \end{bmatrix}^\top$, *is*

$$\boldsymbol{x}_t^\star = \underbrace{\frac{1}{\gamma}\begin{bmatrix} \frac{\lambda_1}{\sigma_1} - \frac{\rho}{\sigma_1\bar{\rho}}\lambda_{2,t} \\ \frac{1}{\sigma_2\bar{\rho}}\lambda_{2,t} \end{bmatrix}}_{\textit{myopic part}} + \underbrace{\frac{1}{\gamma}\begin{bmatrix} 0 \\ \frac{\beta}{\bar{\rho}} \frac{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)[A_1(s-t)\lambda_{2,t} + A_2(s-t)]ds}{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)ds} \end{bmatrix}}_{\textit{intertemporal part}}. \tag{40}$$

Proof The optimal consumption-wealth ratio follows from (30) and (32). The optimal portfolio plan can be obtained by equating the diffusion terms in (11) and (35), the two characterizations of $\mathcal{W}_t$. ∎

For the optimal consumption-wealth ratio, note that it changes with the commodity price, but not with the stock price.

The myopic allocation in the portfolio plan is the same as that in the terminal wealth case. It is a natural outcome when considering that in the analogy of the investor with utility over consumption to a series of investors concerned with terminal wealth, each of the auxiliary investors has identical myopic allocation.

The only new element arising from assuming intermediate consumption is contained in the intertemporal allocation to the commodity. Comparing (40) and (23), it is clear that the intertemporal allocation in the intermediate consumption case is a weighted average of that in the terminal wealth case, using $\mathcal{H}(\lambda_{2,t}, \tau)$ as the weight. For $\mathcal{H}$, (39) implies

$$\frac{\mathcal{W}_t}{c_t^\star} = \int_t^T \mathcal{H}(\lambda_{2,t}, s - t) ds.$$

Hence, $\mathcal{H}(\lambda_{2,t}, \tau)$ can be interpreted as the time-t value of future consumption in τ periods normalized by the optimal consumption rate at time-t. In all, the intertemporal allocation assuming intermediate consumption is an average of those assuming terminal wealth, and the average is weighted by the value of future consumption in each period.

4.3 Welfare analysis

When people are concerned with interim consumption, the utility loss of leaving out the commodity is as follows.

Proposition 4.3 *Suppose that in the Bond-Stock-Commodity economy, an investor is concerned with maximizing the expected utility over life-time consumption as described in (13). If the commodity is excluded in consumption and portfolio decisions, then $\mathcal{L}$ percent of extra initial wealth is needed to achieve the same expected utility level as is obtained by following the optimal strategy given by Theorem 4.2, and*

$$\frac{\mathcal{L}}{100} = \left[\frac{\omega \int_0^T \mathcal{H}(\lambda_{2,0}, s) ds}{1 - e^{-\omega T}} \right]^{\frac{\gamma}{1-\gamma}} - 1. \tag{41}$$

where

$$\omega := \frac{\eta - r(1-\gamma)}{\gamma} - \frac{1}{2} \frac{1-\gamma}{\gamma^2} \lambda_1^2.$$

Proof By reasoning similar to that in the terminal wealth case, the indirect utility function assuming intermediate consumption, denoted by $\mathcal{J}(\mathcal{W}_t, \lambda_{2,t}, t)$, is given by

$$\mathcal{J}(\mathcal{W}_t, \lambda_{2,t}, t) = \frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{-\eta t} \left[\int_t^T \mathcal{H}(\lambda_{2,t}, s - t) ds \right]^\gamma .$$

In the standard Black-Scholes economy after dropping the commodity, the corresponding indirect utility is

$$\mathcal{J}^{BS}(\mathcal{W}_t, t) = \frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} \left[\frac{1}{\omega} \left(1 - e^{-\omega(T-t)} \right) \right]^\gamma ,$$

Then $\mathcal{L}$ as given by (41) follows from its definition. ∎

5 Calibration and discussion

In this section, I shall first estimate the commodity futures price model. Then some representative numerical illustrations and discussions will be presented.

5.1 Estimation of the commodity futures model

The parameters that characterize the asset class of commodities are estimated using the GSCI Commodities Index futures prices. The GSCI index underlying the futures contract tracks the price levels of major commodities, and the futures contract can be viewed as being written on a basket of commodities. Therefore, the GSCI Commodities Index is taken to be the non-tradable spot commodity price, and the GSCI Commodities Index futures prices are the tradable futures prices.

The estimation is carried out in two steps. In the first, the three parameters that characterize the spot commodity price, θ, μ_2, and σ_2, are estimated. From (3), the logarithm of the spot commodity price follows a first-order autoregressive process, and the maximum likelihood method is used to get the estimates (Table 1). I use the monthly data of GSCI Commodity Index from December 1969, the start date of the index, to November 2008. The data are deflated by the US CPI-U index, for the reason that the asset prices are assumed to be measured in real terms in the Bond-Stock-Commodity economy. The deflated data are normalized in such a way that the value was 100 in July 1992 when Chicago Mercantile Exchange (CME) introduced futures on this index.

The second step is to estimate α and β, which specify the risk premium in the futures market (4). The futures price (5) can, in log form, be rewritten as

$$\ln F_t(T) = e^{-\widetilde{\theta}T} m_t + (1 - e^{-\widetilde{\theta}T})\frac{\widetilde{\mu}_2}{\widetilde{\theta}} + \frac{\sigma_2^2}{4\widetilde{\theta}}(1 - e^{-2\widetilde{\theta}T}).$$

Then the system of estimation equations is

$$\boldsymbol{y}_t = \boldsymbol{a} m_t + \boldsymbol{d} + \boldsymbol{\varepsilon}_t, \qquad t = 1, 2, \ldots, K \tag{42}$$

where K is the number of the observations. In the above, $\boldsymbol{\varepsilon}_t$ is an $N \times 1$ vector of disturbance,

$$\boldsymbol{y}_t = \begin{bmatrix} \ln F_t(\mathcal{T}_1) \\ \vdots \\ \ln F_t(\mathcal{T}_N) \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} e^{-\widetilde{\theta}\mathcal{T}_1} \\ \vdots \\ e^{-\widetilde{\theta}\mathcal{T}_1} \end{bmatrix}$$

and

$$\boldsymbol{d} = \begin{bmatrix} \left(1 - e^{-\widetilde{\theta}\mathcal{T}_1}\right)\frac{\widetilde{\mu}_2}{\widetilde{\theta}} + \frac{\sigma_2^2}{4\widetilde{\theta}}\left(1 - e^{-2\widetilde{\theta}\mathcal{T}_1}\right) \\ \vdots \\ \left(1 - e^{-\widetilde{\theta}\mathcal{T}_N}\right)\frac{\widetilde{\mu}_2}{\widetilde{\theta}} + \frac{\sigma_2^2}{4\widetilde{\theta}}\left(1 - e^{-2\widetilde{\theta}\mathcal{T}_N}\right) \end{bmatrix}$$

where $\mathcal{T}_1, \ldots, \mathcal{T}_N$ are the time to maturity of the futures contracts and N is the number of contracts. In view of the liquidity of futures contracts, the first three nearby futures contracts are used for the estimation. In particular, the data, obtained from Bloomberg, consist of the monthly observations of the prices of these three contracts at the end of each month from July 1992 to November 2008. Since the futures trading terminates on the eleventh business day of the contract month, the time to maturity for each of these three contracts does not change with the observations. As with the underlying index, the futures prices are deflated by the CPI-U index, and normalized correspondingly.

The estimates of α and β, presented in Table 1, are obtained by applying the iterative least squares methods to (42). The estimate of β is significantly negative, suggesting that the risk premium of the commodity futures is decreasing in the spot commodity price.

Estimated parameter values					Assumed parameter values				
$\hat{\theta}$	$\hat{\mu}_2$	$\hat{\sigma}_2$	$\hat{\alpha}$	$\hat{\beta}$	μ_1	σ_1	r	ρ	η
0.120	5.023	0.199	3.319	-0.674	0.080	0.150	0.010	-0.100	0.010
(0.082)	(0.066)	(0.012)	(0.323)	(0.069)					

Table 1: **The parameter values used for the numerical exercise** The parameters characterizing the asset class of commodities are estimated, and the standard errors are in parenthesis. The other parameter values are taken to be consistent with many existing studies.

For the purpose of numerical illustration, I assume the other parameter values as given in Table 1. The parameter values for the stock and the riskless bond, μ_1, σ_1, and r, are taken to be consistent with many empirical studies, e.g. Campbell [2003]. The value of the correlation coefficient is chosen to be -0.10, a level corresponding to the finding in many studies that the commodity and stock returns have a moderate negative correlation.

5.2 Optimal strategy and utility of commodity investment

For the parameter values as given in Table 1, Optimal financial strategies and utility losses are determined. The focus of this numerical exercise is on the influence of the commodity price, of the horizon of the investor, and of the risk aversion of the investor. From the property of Ornstein-Uhlenbeck processes, it follows that the logarithm of commodity price is asymptotically stationary, and the asymptotically stationary distribution is normal: $\Phi\left(\mu_2 - \frac{\sigma_2^2}{2\theta}, \frac{\sigma_2^2}{2\theta}\right)$. To have an intuitive idea of the level of current commodity price, I shall locate it with respect to this asymptotical distribution.

Figure 2 shows the optimal strategies and utility losses for a range of current commodity prices from 66 through 251. With respect to the asymptotically stationary distribution of the commodity price, this commodity price range corresponds to the one from the 5th-percentile to the 95th-percentile. In this example, the optimal stock weight is decreasing with the spot commodity price, owing to the assumed negative correlation coefficient between the two driving Brownian motions, and the decreasing relationship between the risk premium in the commodity futures market and the spot commodity price. Notably, the commodity allocation varies considerably with the changes of the commodity price. The commodity allocation decreases from a long position of about 100% for the commodity price at 5th-percentile, to a short position of around 50% for the commodity price at 95th-percentile. In the optimal portfolio strategy, there is substantial market timing.

We have learned that the utility losses of leaving out the commodity depend on the current commodity price (Propositions 3.3 and 4.3). In this numerical example, the current commodity price has a significant impact on the magnitude of utility loss. For the given range of commodity price, the utility loss varies from 10% to 47% in the terminal wealth case, and from 5% to 20% in the intermediate consumption case. It suggests that excluding commodities in financial decisions is much more costly when commodity prices are very low or very high than when they are moderate.

Figure 3 shows that the utility loss of excluding the commodity is increasing in the horizon, T, and decreasing in the degree of risk aversion, γ. The decreasing

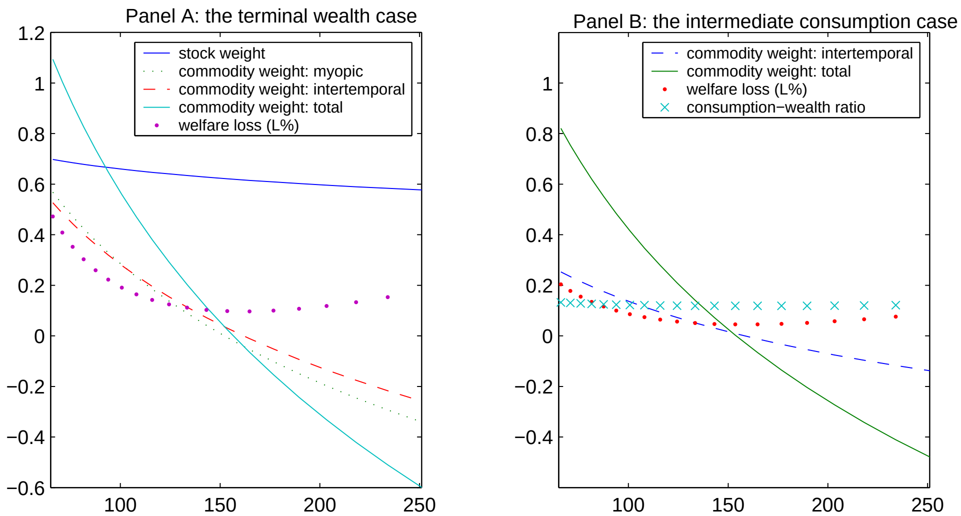

Figure 2: **Optimal policy and the commodity price** This figure shows how the optimal strategy and the utility loss changes with the current commodity price in the terminal wealth case (Panel A), and in the intermediate consumption case (Panel B). It is assumed that $T = 10$, and $\gamma = 5$. The stock allocation and the myopic allocation to the commodity are the same in both cases, and they are not repeated in Panel B for ease of reading.

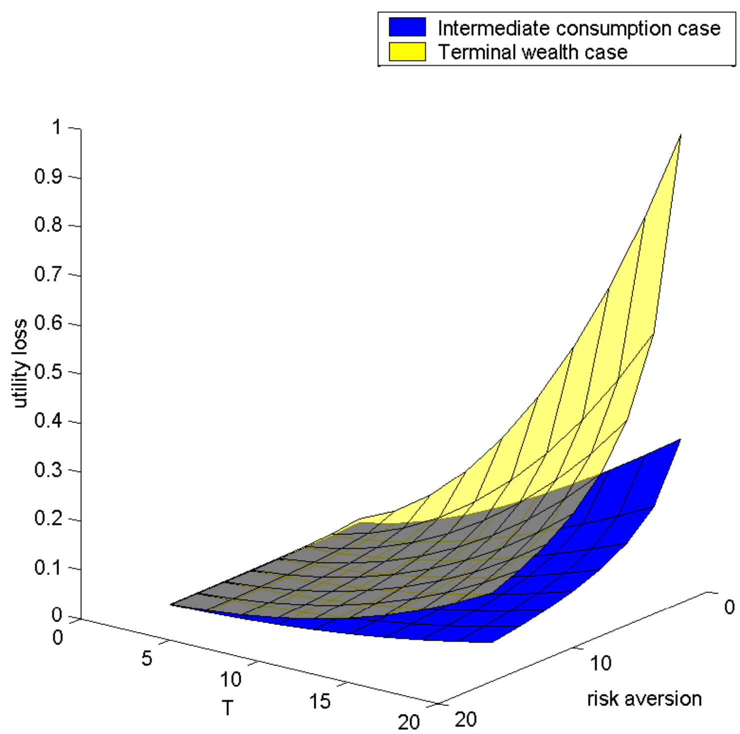

Figure 3: **The effect on utility loss of investment horizon and risk aversion** This figure shows how the utility loss changes with the investment horizon and the degree of risk aversion in the terminal wealth case and the intermediate consumption case. The spot commodity price is assumed to be 129, the median value of the asymptotic distribution.

relation to the degree of risk aversion is perhaps what one would expect, considering that risky assets, including commodities, account for a lower portfolio weight for more risk averse people as they would invest more in the riskless asset. The increasing relation to the horizon also should come as no surprise if considering that a longer period of time enables one to benefit more from the additional investment opportunities brought by commodities. These intuitions make it tempting to conjecture that these relations hold independent of the chosen parameter values.

Figure 3 also shows that other things being equal, excluding the commodity is less costly in the intermediate consumption case than in the terminal wealth case. If other things being equal, the utility cost of excluding the commodity is increasing in the horizon, then a plausible explanation for the lower utility cost in the intermediate consumption case could be that the "effective" horizon of the investor assuming intermediate consumption is shorter than that assuming terminal wealth. The aforementioned analogy to coupon-bearing and zero-coupon bonds helps to understand the explanation. It is well known that a coupon-bearing bond has a shorter effective horizon than its zero-coupon counterpart. By analogy, the investor assuming intermediate consumption should have a shorter effective horizon than her counterpart assuming terminal wealth, and hence suffers a lower utility cost.

The finding that the utility loss is decreasing in risk aversion is different from that of Anson [1999], who concludes that the more risk-averse the investor, the higher the utility of investing in commodity futures. A plausible explanation of this difference lies in the different framework of analysis: Anson's study is based on a one-period mean-variance framework with only risky assets, while our conclusion comes from a dynamic framework where the riskless asset is available. Anson [1999] shows that the allocation to commodity futures is monotonically increasing with the degree of risk aversion. In comparison, this numerical exercise shows the effect of risk aversion on commodity allocation is not monotonic (Figure 4). We can distinguish two cases: (i) when the spot commodity price is relatively low, the commodity weight is decreasing with risk aversion; and (ii) when the spot commodity price is relatively high, the commodity weight is increasing with risk aversion.

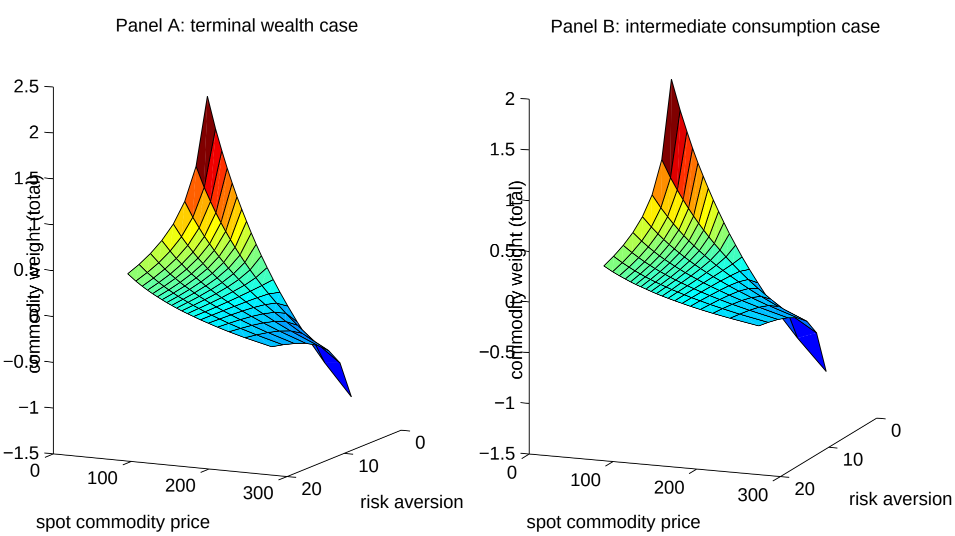

Figure 4: **The effect on the optimal commodity weight of the spot commodity price and risk aversion** This figure shows how the total commodity weight changes with the spot commodity price and the degree of risk aversion in the terminal wealth case and the intermediate consumption case. It is assumed that $T = 10$.

Now turn to the intertemporal allocation to the commodity. Figure 5 shows how the intertemporal commodity allocation changes with the investment horizon and the degree of risk aversion. Two interesting points stand out from this figure: (i) in both terminal wealth and intermediate consumption cases, the commodity allocation increases with the horizon of the investor; and (ii) the investor with utility over terminal wealth allocates more to the commodity than the investor concerned with intermediate consumption. Actually, this figure illustrates the following proposition.

Proposition 5.1 *If $\lambda_{2,t}$ and λ^* are positive, then other things being equal, the optimal commodity allocation increases with the horizon in both the terminal wealth case and the intermediate consumption case. Moreover, the optimal intertemporal allocation to the commodity is greater in the terminal wealth case than in the intermediate consumption case.*

Proof When λ^* is positive, then $A_1(\tau)$, $A_2(\tau)$ and their derivatives are all negative (see Appendix A.3). This implies that

$$\begin{aligned} -A_1(T-t) &> -\frac{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)A_1(s-t)ds}{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)ds} \\ -A_2(T-t) &> -\frac{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)A_2(s-t)ds}{\int_t^T \mathcal{H}(\lambda_{2,t}, s-t)ds}. \end{aligned}$$

From (23) and (40) and condition that $\lambda_{2,t}$ is positive, the statement about the intertemporal allocation follows. ∎

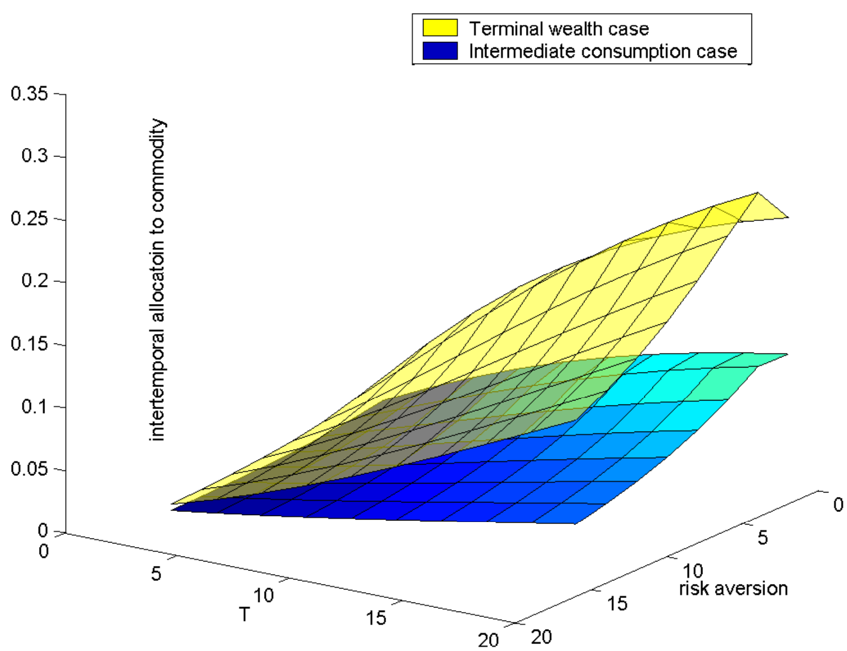

Figure 5: **The effect of horizon and risk aversion on the intertemporal commodity allocation** This figure shows how the intertemporal commodity allocation changes with the horizon and the degree of risk aversion in the terminal wealth case and the intermediate consumption case. The spot commodity price is assumed to be 129, the median value of the asymptotic distribution.

Note that given the parameter values and the current commodity prices used in Figure 5, λ^* and $\lambda_{2,t}$ are positive.

In sharp contrast to the significant utility of commodity investment in the dynamic framework, the commodity market in this example is of little utility for a one-period mean-variance investor. With these parameter values, the unconditional Sharpe ratio in the commodity market, $E[\lambda_{M,t}]$, is 6.5%. In comparison, the Sharpe ratio in the stock market is 47%. Given an empirically reasonable value of the correlation coefficient, the allocation to and utility of the commodity investment in a mean-variance optimized portfolio would be of little practical relevance. Therefore, investing in this asset class makes little sense in the one-period mean-variance optimization framework. But under the dynamic framework used in this study, commodity allocation may be needed in considerable magnitude, and allocation to the asset class of commodities may yield a consequential welfare improvement. The difference in the perception of the asset class of commodities stems from the difference in the research frameworks.

The large utility of commodity investment illustrated in this example relies on exploiting the return predictability in the commodity market. In this study, however, I abstract from robustness issues, especially uncertainty about the predictive relationship, which would affect the optimal strategy and the utility improvement arising from commodity investment. Further research is needed to address these issues.

6 Conclusion

Given the rapid growth of commodity markets as a major alternative asset class, this study has aimed at examining this asset class in the framework of dynamic asset allocation à la Merton [1969]. In this study, the risk premium in the commodity market is assumed to be time-varying and dependent on the non-tradable spot commodity price, while the spot commodity price has the property of mean reversion. This characterization of the commodity market, an extended version of the single-factor model in Schwartz [1997], is consistent with the empirical finding in this paper that the spot commodity price has significantly negative effect on the risk premium. As such, this study underscores the implications for portfolio and consumption decisions of the time-varying risk premia of commodity investment arising from the mean-reverting nature of commodity prices.

I have derived the optimal dynamic strategies for an investor who is concerned either with terminal wealth or with intermediate consumption. Closed-form expressions were obtained for the optimal strategies and the utility losses of excluding the commodity in financial decisions. In the optimal policy, the allocation to commodities is made for myopic purposes, as well as for intertemporal hedging purposes. The optimal allocation to the stock is solely myopic, and dependent on the spot commodity price.

Based on some representative parameter values, I presented a numerical example for the optimal strategy and the utility cost of leaving out the commodity. In this example, as long as the risk premium in the commodity market is positive, the optimal commodity allocation has a horizon effect—the longer horizon, the more allocation to the commodity. The example also shows that if the investor is relatively less risk averse, and concerned with financial decisions over a longer time period, then the commodity market brings a greater utility improvement. It suggests that in a long-term financial plan, like that of saving for retirement, it is rather costly to exclude commodities from financial decision making.

In this example, the significant utility of commodity investment is established from the dynamic perspective on financial decisions, while an investor from the static mean-variance perspective would find little value in commodity investment. In the static mean-variance framework, an appreciably positive risk premium in commodity markets is needed for investors to include commodities in mainstream portfolios as an appealing asset class. When dynamic investment strategies are allowed, however, considerable utility of commodity investment may come from exploiting the return predictability in commodity markets.

A Appendix

A.1 The portfolio strategy of the representative commodity portfolio $\mathcal{M}_t$

Applying Ito's formula to the futures price (5) yields

$$dF_t(T) = F_t(T)e^{-\widetilde{\theta}(T-t)}\sigma_2\lambda_{M,t}dt + F_t(T)e^{-\widetilde{\theta}(T-t)}\sigma_2 dZ_{M,t}.$$

Consider a self-financing portfolio $\mathcal{M}_t$ consisting of ψ_t units of the riskless bond and ϕ_t units of the futures contract. Because the futures contract is constantly resettled to have zero value and the portfolio is self-financing, the portfolio $\mathcal{M}_t$ satisfies

$$\mathcal{M}_t = \psi_t B_t, \quad \text{and} \quad d\mathcal{M}_t = \psi_t dB_t + \phi_t dF_t(T).$$

If the units of the futures contract $\phi_t = \frac{\mathcal{M}_t}{F_t(T)}e^{\widetilde{\theta}(T-t)}$, then

$$d\mathcal{M}_t = \mathcal{M}_t\left(r + \sigma_2\lambda_{M,t}\right)dt + \mathcal{M}_t\sigma_2 dZ_{M,t}.$$

Note that the above equation only holds for $t \leq T$. To overcome this restriction, we can consider rolling over the futures contract. Namely, the holding of futures contract is rolled over to keep the time to maturity of the held contract equal to a constant ℓ, and the units of futures contract $\phi_t = \frac{\mathcal{M}_t}{F_t(T)}e^{\widetilde{\theta}\ell}$. Under this portfolio strategy, the notional value of the futures holding is

$$\phi_t F_t(t+\ell) = \mathcal{M}_t e^{\widetilde{\theta}\ell}.$$

A.2 Validity of the optimal policy

This appendix shows the validity of the optimal strategies that have been derived in the terminal wealth case and in the interim consumption case. In addition to the assumption that $E(\xi_T^{-1})$ is finite, the following two conditions need to be verified [Cox and Huang, 1989, Theorem 2.2]: (i) the Lagrange multipliers k and $\mathcal{K}$ are positive and finite; and (ii) F and $\mathcal{G}$ have sufficient continuous differentiability.

The Lagrange multipliers k and $\mathcal{K}$ are positive and finite From (10), it follows that

$$\begin{aligned}\xi_T &= \exp\left[-\left(r + \frac{1}{2}\lambda_1^2\right)T - \lambda_1 Z_{1,T} - \frac{1}{2}\int_0^T \lambda_{2,t}^2 dt - \int_0^T \lambda_{2,t} dZ_{2,t}\right] \\ &\leq \exp\left[-\left(r + \frac{1}{2}\lambda_1^2\right)T - \lambda_1 Z_{1,T} - \int_0^T \lambda_{2,t} dZ_{2,t}\right].\end{aligned}$$

It can be seen that ξ_T is positive. Moreover, because the right hand side is lognormal, $E[\xi_T]$ is bounded by the lognormal variable's expectation, which is finite and a continuous function of time. The two Lagrange multipliers are given by

$$\begin{aligned} k &= W_0^{-\gamma}\left(E\left[\xi_T^{1-1/\gamma}\right]\right)^{\gamma} \\ \mathcal{K} &= W_0^{-\gamma}\left(E\left[\int_0^T \xi_t^{1-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta t} dt\right]\right)^{\gamma}. \end{aligned}$$

Their positiveness follows immediately, and their finiteness follows from Jensen's inequality and the assumption that $\gamma > 1$.

Continuous differentiability of F and $\mathcal{G}$ From (17) and (32), the differentiability condition holds inasmuch as $A_1(\tau)$, $A_2(\tau)$, $A_3(\tau)$ and $\mathcal{A}_3(\tau)$ are continuously differentiable. From (19) and (34), it follows that they are indeed continuously differentiable (for the continuous differentiability of $A_1(\tau)$ and $A_1(\tau)$, see (43) below).

A.3 Properties of $A_1(\tau)$ and $A_2(\tau)$

Define

$$B(\tau) := 2q - (q + a_1)\left(1 - e^{-q\tau}\right).$$

Because $q > 0$, and $q > a_1$, $B(\tau) > 0$. From the assumption that $\gamma > 1$, it follows that

$$\begin{aligned} A_1(\tau) &< 0, \\ \text{sign}\big(A_2(\tau)\big) &= -\text{sign}(\lambda^*). \end{aligned}$$

The derivatives of $A_1(\tau)$ and $A_2(\tau)$ are

$$\begin{aligned} A_1'(\tau) &= \frac{1-\gamma}{\gamma}\frac{4q^2 e^{-q\tau}}{B(\tau)^2} \\ A_2'(\tau) &= \frac{1-\gamma}{\gamma}\frac{4\lambda^*\left(e^{-q\tau/2} - e^{-q\tau}\right)\left[q - a_1 + (q + a_1)e^{-q\tau/2}\right]}{B(\tau)^2} \end{aligned} \tag{43}$$

So $A_1'(\tau)$ and $A_2'(\tau)$ are continuous, and their signs are

$$\begin{aligned} A_1'(\tau) &< 0, \\ \text{sign}\left(A_2'(\tau)\right) &= -\text{sign}(\lambda^*). \end{aligned}$$

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