Robust Portfolio Solutions to Mean-Variance (MV) Portfolio Selections

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OUTLINE

- Impact of estimation error in MV portfolio optimization
- Min-max robust MV portfolio optimization
- CVaR robust MV portfolio optimization
  - Estimation risk measured by CVaR
  - Properties of CVaR robust portfolios
- Concluding remarks

Mean-Variance Optimization: Markowitz, 1950

$$\min_{x \in \Re^n} -\mu^T x + \lambda \cdot x^T Q x$$

$$\text{s.t.} \quad e^T x = 1$$

$$x \geq 0$$

μ: expected rate of return

Q: covariance matrix

x: percentage portfolio holdings

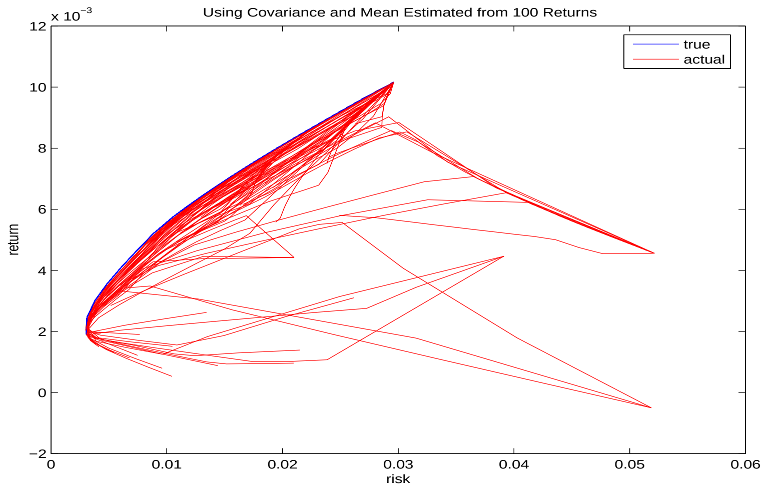
$\lambda \geq 0$: the risk aversion parameter

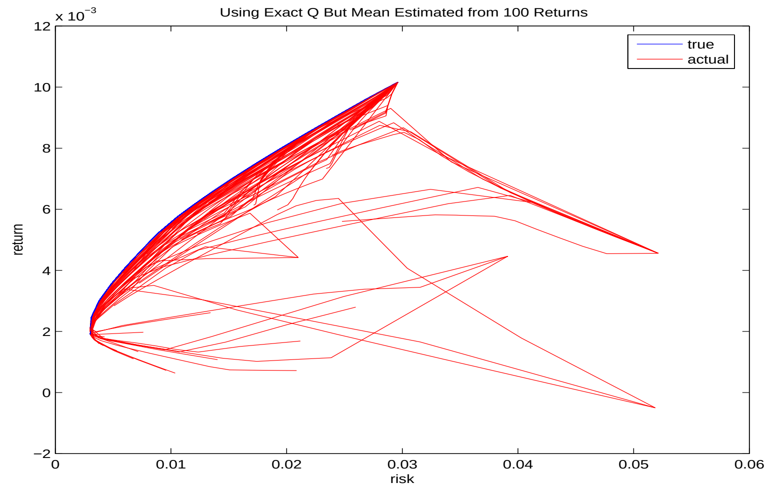
e^T: $[1, 1, \cdots, 1]$

- The curve $(\sqrt{x(\lambda)^T Q x(\lambda)}, \mu^T x(\lambda)),\ \lambda \geq 0$, forms an efficient frontier, where $x(\lambda)$ denotes the optimal MV portfolio for λ.

- In practice, only estimates $\tilde{\mu}, \tilde{Q}$ from a finite set of return samples are available. MV QP with estimates $\tilde{\mu}$ and $\tilde{Q}$ is a nominal MV problem.

- Actual frontier (Broadie, 1993): calculated using the estimated efficient portfolios and the true parameter values
$(\sqrt{x(\lambda)^T Q x(\lambda)}, \mu^T x(\lambda)), \lambda \geq 0$





- True efficient frontier: computed using μ and Q
- Actual frontier: computed based estimates $\tilde{\mu}$ and $\tilde{Q}$ using 100 return samples
- Maximum return portfolio: concentrates on a single asset
- Optimal portfolio $\tilde{x}(\lambda)$ from estimates $\tilde{\mu}, \tilde{Q}$ may not perform well in reality.

Estimation Error in Mean Returns

- Given data in a fixed time period, increasing frequency of the observation does not increase the accuracy in the estimate of the mean.
- Increasing the frequency of observation does improve the estimate of the volatility (Merton 1980).

In this talk, we focus on uncertainty in mean returns and assume covariance Q is given.

Examples of research addressing estimation error in MV optimization:

- Incorporating additional views: Black-Litterman, 1992
- Resampling approach: Michaud (1998), Meucci (2005), $\cdots$
- Baysian approach: Zellner and Chetty (1965), Brown (1976), $\cdots$
- Robust optimziation: Goldfarb and Iyengar (2003), Tütüncü and Koenig (2003), Garlappi, Uppal and Wang (2007)

What About Min-Max Robust Solutions?

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu \in \mathcal{S}_\mu, Q \in \mathcal{S}_Q} -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \ . \end{array}$$

- typically a semi-infinite programming problem: computationally challenging
- $\mathcal{S}_\mu, \mathcal{S}_Q$: uncertainty sets for μ and Q

When uncertainty sets are

- interval uncertainty set: $\mu_L \leq \mu \leq \mu_R$
- ellipsoidal uncertainty set: $(\bar{\mu} - \mu)^T A(\bar{\mu} - \mu) \leq \chi$

the problem (a semi-definite programming problem) is easy to solve.

Specification of uncertainty sets plays crucial roles in robust solutions.

Garlappi, Uppal, Wang (2007) derive an explicit formula for the min-max robust solution using the ellipsoidal uncertainty set for μ, assuming Q is known and short selling is allowed.

Their uncertainty set is based on the following statistical result:

Assume that n asset returns have a joint normal distribution and mean estimate $\bar{\mu}$ is computed from T samples. If the covariance matrix Q is known, the quantity

$$\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$$

has a χ_n^2 distribution with n degrees of freedom.

Garlappi, Uppal, Wang (2007) derive the analytic solution to:

$$\begin{array}{ll} \min_x & \max_\mu -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & (\bar{\mu} - \mu)^T Q^{-1} (\bar{\mu} - \mu) \leq \chi \\ & e^T x = 1 \end{array}$$

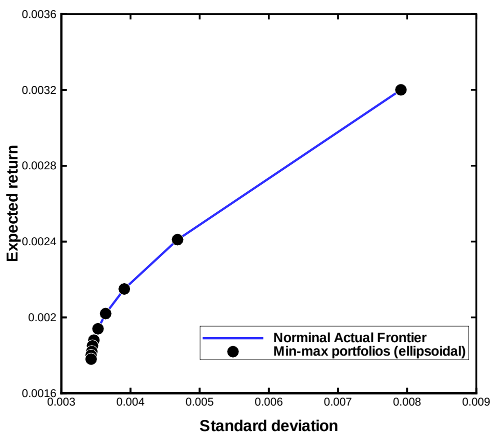
We show [a] that, even with additional $x \geq 0$, the solution to the (above) min-max robust problem is a solution to the nominal problem

$$\begin{array}{ll} \min_x & -\bar{\mu}^T x + \hat{\lambda} \cdot x^T Q x \\ \text{subject to} & e^T x = 1, \quad x \geq 0, \end{array}$$

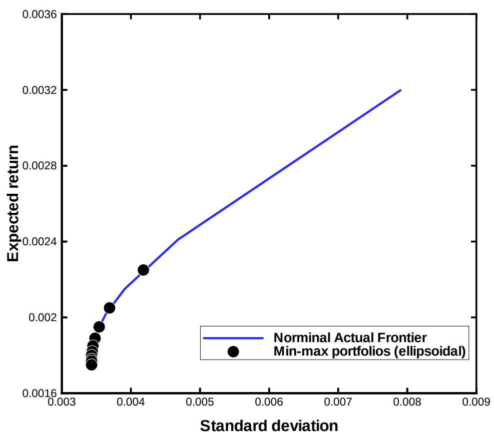
with $\hat{\lambda} \geq \lambda$.

[a] L. Zhu, T. F. Coleman, and Y. Li, *Minmax robust and CVaR robust mean variance portfolios*, 2008. To appear in *Journal of Risk*.. See also Schöttle, K., R. Werner. 2006. Towards reliable efficient frontiers. *Journal of Asset Management* **7** 128–141 and Meucci, A. 2005, *Risk and Asset Allocation*. Springer.

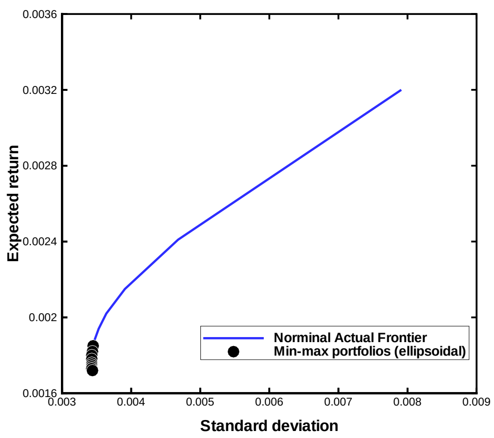
Min-max Robust Frontier vs Nominal Actual Frontier



(a) $\chi = 0$



(b) $\chi = 5$



(c) $\chi = 50$

Observations:

- Minmax robust (actual) frontier is a squeezed segment of the frontier of the nominal problem from the estimate.
- Estimation risk is dealt with in the same way as the return risk.

Interval uncertainty set: $\mu_L \leq \mu \leq \mu_R$

$$\begin{array}{ll} \min_{x} & \max_{\mu_L \leq \mu \leq \mu_R} -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \ . \end{array}$$

The robust solution solves

$$\begin{array}{ll} \min_{x} & -\mu_L{}^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \ . \end{array}$$

$\Longrightarrow$ Minmax robust portfolio depends on μ_L

$\Longrightarrow$ Minmax return portfolio ($\lambda = 0$) concentrates on a single asset

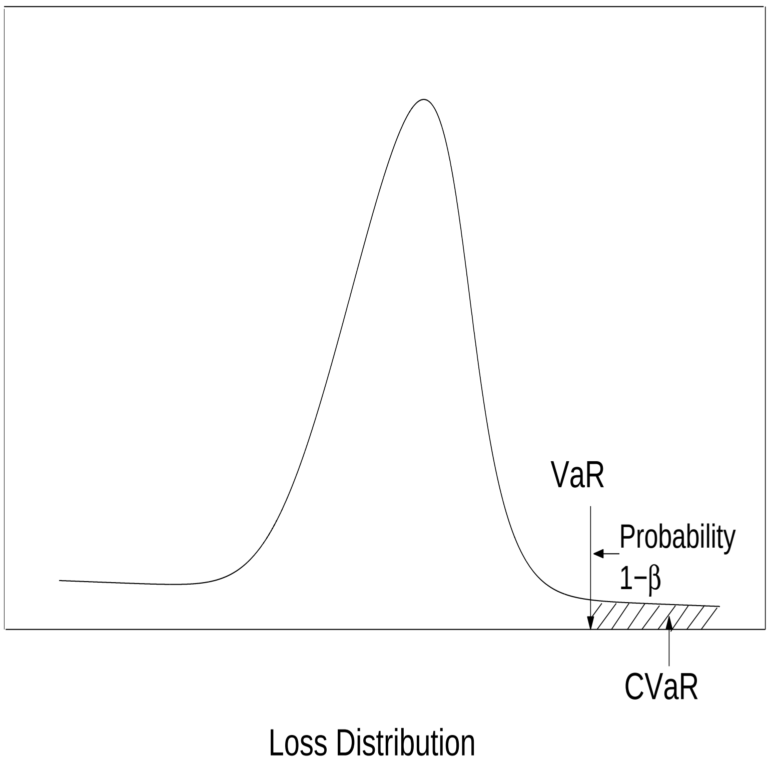
We can regard uncertainty in parameter μ as an estimation risk.

To measure the estimation risk, use the statistical result for the mean estimates that

$$\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$$

has a χ_n^2 distribution with n degrees of freedom.

Measure Tail Estimation Risk: CVaR



β: confidence level, e.g., $\beta = 90\%$

CVaR-Robust Mean Variance Portfolio

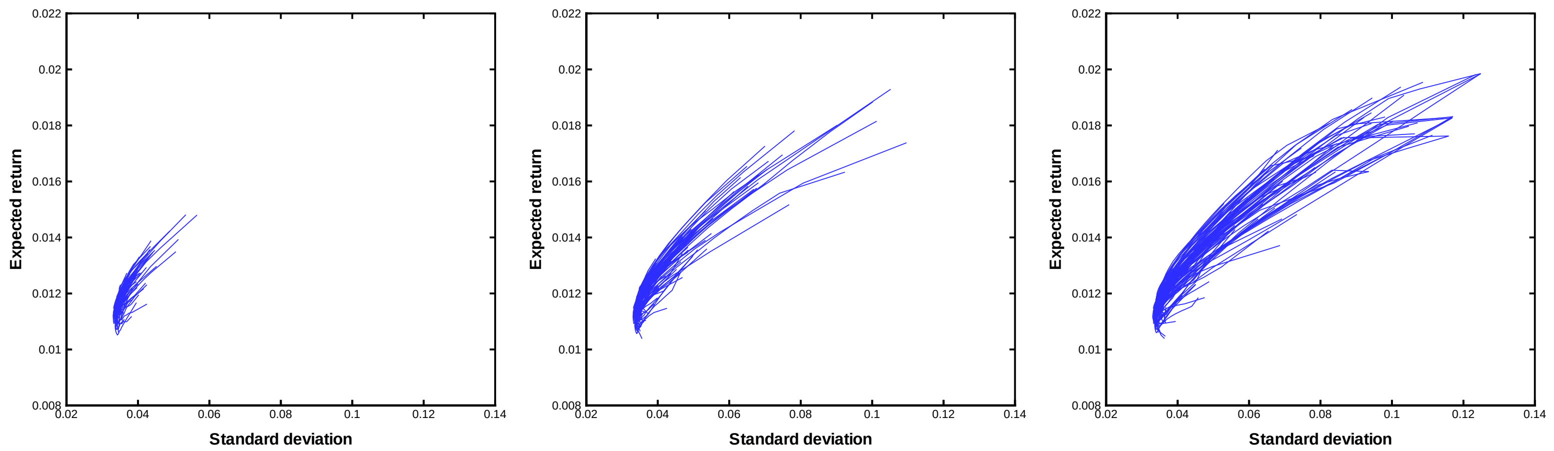
$$\begin{array}{ll} \min\limits_{x} & \text{CVaR}_{\beta}^{\mu}(-\mu^T x) + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \end{array}$$

Statistics:

$$\frac{T(T-n)}{(T-1)n}(\bar{\mu} - \mu)^T Q^{-1}(\bar{\mu} - \mu)$$

has a χ_n^2 distribution with n degrees of freedom.

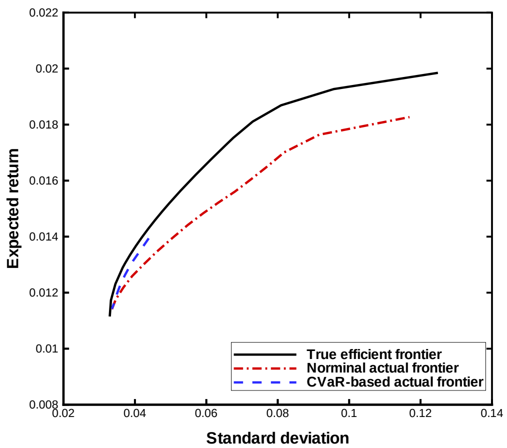
CVaR Robust Actual Frontiers



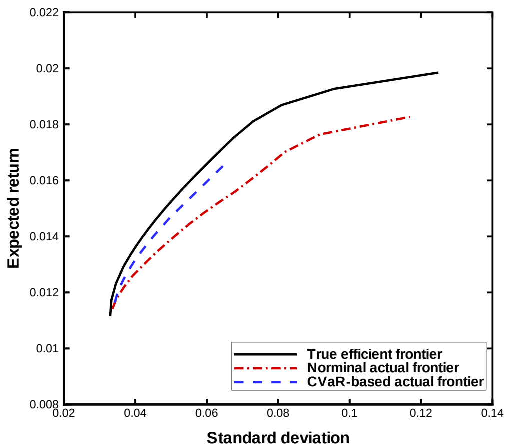
(d) β: 90% (e) β: 60% (f) β: 30%

$\bar{\mu}$ is estimated from 100 return samples; 10,000 MC samples for μ

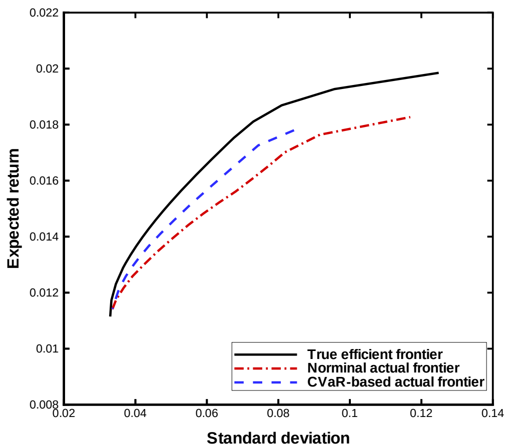
Frontiers of CVaR robust portfolios are different from frontiers of portfolios from nominal estimates.



(g) $\beta = 90\%$



(h) $\beta = 60\%$

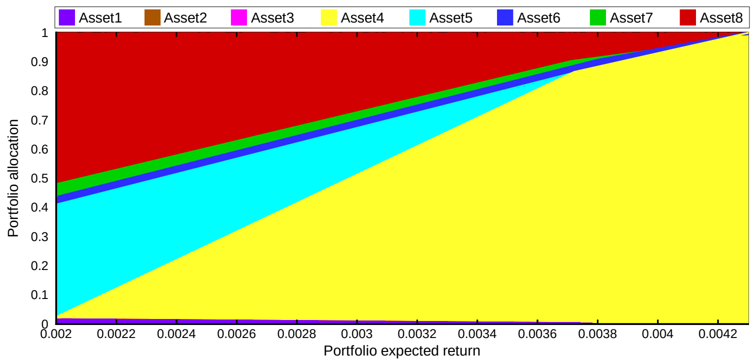


(i) $\beta = 30\%$

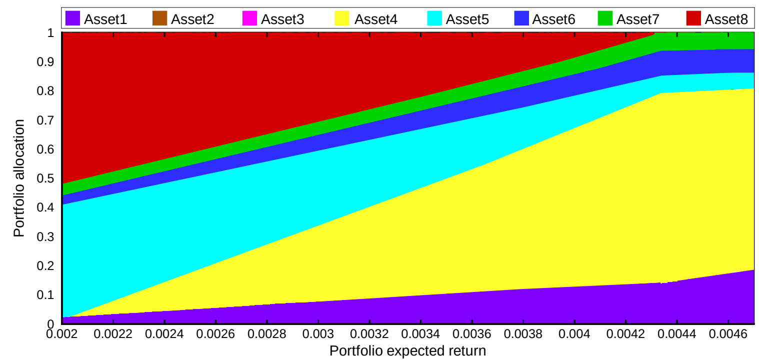
The confidence level β can be interpreted as an estimation risk aversion parameter:

- As $\beta \Longrightarrow 1$, extreme loss due to uncertainty in μ is emphasized. This corresponds to increasingly strong aversion to estimation risk.
- As $\beta \Longrightarrow 0$, average loss due to uncertainty in μ is considered. This corresponds to increasing tolerance to estimation risk.

Composition Comparison: Min-max Robust and CVaR Robust



(j) Min-max robust ($\mu_L \leq \mu \leq \mu_R$) portfolio



(k) CVaR robust (90%) portfolios

For CVaR robust formulation,

- The maximum return portfolio are often diversified.

Computing CVaR Robust Portfolios

- Computing a CVaR Robust Portfolio by Solving a QP Based on MC
  - $O(m+n)$ variables and $O(m+n)$ constraints, e.g., $n = 100$, $m = 10,000$
  - $n \times m$ dense block in the QP
  - Computational cost can become large as m and n become large.
- We propose to compute CVaR robust portfolios using a smoothing technique [a] $\Longrightarrow$ solving a smooth optimization problem with
  - $O(n)$ variables and $O(n)$ constraints

[a] Xu, H., D. Zhang. 2008. Smooth sample average approximation of stationary points in nonsmooth stochastic optimization and applications. *Math. Programming., Ser. A.* .

CPU Comparisons

	MOSEK (CPU sec)			Smoothing (CPU sec)		
# samples	8 assets	50 assets	148 assets	8 assets	50 assets	148 assets
5000	0.39	1.75	7.06	0.42	0.34	1.98
10,000	0.77	4.25	10.38	0.75	0.50	4.13
25,000	2.56	10.83	34.97	1.77	1.36	10.25

$$\lambda = 0, \ \beta = 90\%$$

Concluding Remarks

When mean return is uncertain for MV portfolio selection,

- Minmax robust with ellipsoidal uncertainty set: squeezed frontier from MV based on nominal estimates
- Minmax robust with interval uncertainty set: the maximum return portfolio is never diversified
- CVaR robust:
  - different frontiers from nominal estimates
  - maximum return portfolios are typically diversified into multiple assets
  - confidence level β can be naturally interpreted as a risk aversion parameter towards estimation risk

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