13th May 2009

Hierarchical Archimedean Copulae over time dependence with applications to Financial Data

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Abstract: In the classical multivariate time series models the residuals are assumed to be normally distributed. However the assumption of normality is rarely consistent with the empirical evidence and leads to possibly incorrect inferences from financial models. The copula theory allows us to extend the classical time series models to nonelliptically distributed residuals. In this paper we analyze the time behavior of hierarchical Archimedean copulas. This class is a generalization of the Archimedean copulas and allows to model more general non-exchangeable dependency structures. In this paper we use hierarchical Archimedean copulae with adaptively estimated time varying parameters and structures for modelling the distribution of returns. We also compare the classical rolling window approach to the local change point detection.

Keywords: copula; multivariate distribution; Archimedean copula; Adaptive Estimation.
JEL Classification: C14.

1 Introduction

The key difference between the modelling of univariate and multivariate observations is the possibility to model the dependencies between the components of a multivariate data set. The form and strength of the dependency is very important in many applications, especially in finance. For example, the diversification, asset allocation, financial spillovers well illustrate the importance of dependencies. The most straightforward and well established in the literature approach to modeling dependencies is via the correlation (or covariance) matrix. The correlation matrix uniquely characterizes the dependency, if the data is driven, for example, by a multivariate normal distribution. In this case, the sample covariance matrix provides us with an efficient and consistent estimator of the true dependencies. Similar arguments also hold for arbitrary elliptical distributions (as multivariate t). Due to its simplicity the covariance matrix became a major model of dependency. There two important drawbacks of this method we discuss next.

The true covariance matrix should not be constant over time. This refers both to the variances and to the correlations. The time varying conditional volatilities are typically modeled using GARCH-type processes. This models are successful in one dimensional framework, however, lacks its simplicity in the multivariate case. There are numerous approaches to modeling time-varying conditional correlations with DCC, CCC, BEKK, among others. For a recent review of multivariate GARCH process we refer to Silvennoinen and Teräsvirta (2009). The estimation of these models requires, however, data sets over long time periods. Therefore, we assume that the parameters of the process are constant over the estimation period. The validity of this assumption is doubtful even in the univariate case, see Lamoureux and Lastrapes (1990). Thus if we use long historical periods to estimate a GARCH process, we obtain potentially biased estimators. Moreover, the GARCH processes introduce the varying dependency directly into the data generating process. An alternative approach is to use regime switching models, see Lange and Rahbek (2009). The disadvantage of this approach is that the regimes of dependency are estimated from past data and for further modeling as given. In practice, however, it is more realistic that the parameters characterizing the dependency change arbitrary. The problem is similar to detecting structural breaks in financial or economic data.

The second disadvantage of the covariance matrix is the fact that it fails to capture many important types of relationships. First, the covariances are measures of linear dependence and generally fail to capture nonlinear relationships appropriately. As an alternative we can consider other measures such as Kendall's τ or Spearman's ρ, see Joe (1997). However, the extensions of these measures to dimensions higher than two is problematic (Schmid and Schmidt (2006)). Second, the elliptical distribution postulate symmetric dependency, i.e. the strength of the relationship is the same for high and low values. This is, however, often a too restrictive assumption. Third, the covariance matrix is usually used as a parameter of the multivariate normal distribution. This assumption is also highly criticized, because it fails to capture heavy tails typical for asset returns. An approach which partially solves these problems is based on copulas, proposed by Sklar and reviewed in Joe (1997) and Nelsen (2006). Copulas allow us to model the dependency separately from the marginal distributions and provide a better fit for heavy tails, asymmetries, etc.

Time-varying copulas were considered recently by Patton (2004), Rodriguez (2007) and Giacomini, Härdle and Spokoiny (2009). Patton (2004) considers an asset-allocation problem with a time-varying parameter of the bivariate copulas. The parameter is estimated as a function of lagged explanatory variables. Rodriguez (2007) studies financial contagion using switching-parameter bivariate copulas. In contrary to these papers Giacomini et al. (2009) use a novel method based on the local adaptive estimation (LAVE) discussed in Spokoiny (2008). The authors consider only a time-varying 6 dimensional simple Archimedean copula. The single parameter of the copula is estimated adaptively. The idea of the approach is to determine the optimal period of time where the parameter of the copula is constant. The shortcoming of the considered approaches is that they are applied either to bivariate or very restrictive multivariate copula models.

The aim of this paper if to develop new models of time-varying dependencies based on copulas with more complex structure, than with just one parameter. We extend the results of Giacomini et al. (2009) to multivariate hierarchical Archimedean copulas which do not suffer from restrictive types of dependencies. A detailed analysis of this class is given in Okhrin, Okhrin and Schmid (2009a) and Okhrin, Okhrin and Schmid (2009b). In contrary to simple Archimedean copulas, the HAC is characterized not only by its parameters, but also by the structure. The time-varying dependency variability of the structure and variability of the parameters. For this reason the approach of Giacomini et al. (2009) cannot be extended directly. The variability of the parameters implies that the dependency becomes stronger or weaker; variability of the structure implies that there is a change not only in the strength of the dependency, but also in its form. The proposed estimation technique allows to determine the periods with permanent structure. The proposed procedure can also be applied to estimate time-varying dependency modeled by a correlation matrix. This gives us a fair method or comparing the periods with constant dependency modeled by copulas and by elliptical distributions.

To assess the performance of the developed methodology we perform an extensive simulation study and empirical study. Within the simulation study we show that the LCP approach quickly reacts to shifts in the structure and in the parameters. This evidence was exploited on real data. To assess the performance of the LCP we compared it with the rolling window estimator using different alternative measures of changes in the structure and in the parameters. We conclude that the LCP approach outperforms the rolling window estimation.

The paper is structured as follows. In the second section wi give a short theoretical background of HAC with estimation and grouping techniques. Section 4 and 5 give a short excursus into the theory of divergencies between distributions and into local adaptive estimation procedures respectively. Section number 6 provide a detailed statistical analysis of the data used in the paper. Sections 7 and 8 deals with applications of the rolling window and LCP to simulated and real data, taken from Datastream.

2 Hierarchical Archimedean Copulae

The advantage of the copula is that it allows to split the multivariate distribution into the margins and pure dependency. It captures the dependency between variables eliminating the impact of the marginal distributions. Formally the copulas where introduced in Sklar (1959). The main results states that if F is an arbitrary d-dimensional continuous distribution function, then the associated copula is unique and defined as a continuous function $C : [0,1]^d \to [0,1]$ which satisfies the equality

$$F(x_1,\dots,x_d) = C\{F_1(x_1),\dots,F_k(x_d)\}, \quad x_1,\dots,x_d \in \mathbb{R},$$

where $F_1(x_1),\dots,F_k(x_d)$ are the respective marginal distributions. Alternatively the copula can be defined as an arbitrary distribution function on $[0,1]^d$ with all margins being uniform. It is clear that for a given distribution its copula can be written as

$$C(u_1,\dots,u_d) = F\{F_1^{-1}(u_1),\dots,F_d^{-1}(u_d)\}, \quad u_1,\dots,u_d \in [0,1],$$

where $F_1^{-1}(\cdot),\dots,F_d^{-1}(\cdot)$ are the corresponding quantile functions. If F belongs to the class of elliptical distributions, then this results in the elliptical copula. Note, however, that this copula cannot be given explicitly, because the distribution function F and the inverse marginal distributions F_i usually have only integral representation. The class of Archimedean copulas overcomes this drawback of the elliptical copulas. Let ϕ denote the Laplace transform of the CDF M of a positive random variable, i.e. $\phi(t) = \int_0^\infty e^{-tw}dM(w)$. For an arbitrary CDF F there exists a unique CDF G, such that

$$F(x) = \int_0^\infty G^\alpha(x)dM(\alpha) = \phi\{-\ln G(x)\}.$$

If F is k-variate with margins $F_1, \dots, F_k$, then

$$F(x_1, \dots, x_d) = \int_0^\infty G_1^\alpha \cdot \dots \cdot G_d^\alpha dM(\alpha) = \phi\left\{-\sum_{i=1}^d \ln G(x_i)\right\} = \phi\left[\sum_{i=1}^d \phi^{-1}\{F_i(x_i)\}\right]. \quad (1)$$

This implies that the d-dimensional copula of F is given by

$$C(u_1, \dots, u_k) = \phi\{\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0, 1], \quad (2)$$

where $\phi \in \mathfrak{L} = \{\phi : [0; \infty) \to [0, 1] \,|\, \phi(0) = 1, \phi(\infty) = 0; (-1)^j \phi^{(j)} \geq 0; j = 1, \dots, \infty\}$. The function ϕ is called the generator of the copula and $\mathfrak{L}$ is the set of Laplace transforms. McNeil and Nešlehová (2008) provide necessary and sufficient conditions on ϕ, which guarantee the feasibility of the Archimedean copula. The generator ϕ is required to be d-monotone, i.e. differentiable up to the order $d-2$, with $(-1)^i \phi^{(i)}(x) \geq 0$, $i = 0, \dots, d-2$ for any $x \in [0, \infty)$ and with $(-1)^{d-2}\phi^{(d-2)}(x)$ being nondecreasing and convex on $[0, \infty)$. For simplicity we make a stronger assumption that ϕ is a completely monotone function, i.e. $(-1)^i \phi^{(i)}(x) \geq 0$ for all $i \geq 0$. The representation of the copula in terms of the Laplace transforms is very useful for simulation purposes (see Whelan (2004), McNeil (2008), Marshall and Olkin (1988)). A detailed review of the properties of Archimedean copulas can be found in McNeil and Nešlehová (2008) and Joe (1996). In general the generator ϕ can depend on several parameters. However, throughout the paper we consider only the generator functions with a single parameter. This is consistent with the literature.

The disadvantage of the Archimedean copulas is the fact that the rendered dependency is symmetric with respect to the permutation of variables, i.e. the distribution is exchangeable. Moreover, the multivariate dependency structure is not flexible, since it depends on a single parameter of the generator function ϕ. To overcome these shortcomings we rely of the link between the Archimedean copulas and the Laplace transforms.

Note that the product copula $G_1^\alpha \cdot \dots \cdot G_d^\alpha$ in (1) can be replaced with an arbitrary multivariate copula $K(G_1^\alpha, \dots, G_d^\alpha)$. Furthermore, the single CDF M can be replaced with an arbitrary d-variate CDF M_d, such that its jth univariate margin has the Laplace transform ϕ_j, $j = 1, \dots, d$. This procedure leads to Hierarchical Archimedean Copulas (Joe (1997)). For example, the copula can be given by

$$C(u_1, \dots, u_d) = \int_0^\infty \dots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1, \alpha_2)\, G_3^{\alpha_2}(u_3) dM_2(\alpha_2, \alpha_3) \dots G_d^{\alpha_{d-1}}(u_d) dM_{d-1}(\alpha_{d-1}).$$

Other orders of integration and combinations of the G_i functions lead to different dependencies. HAC are thoroughly analyzed in Joe (1997), Whelan (2004), Savu and Trede (2006), Embrechts et. al. (2002), McNeil (2008).

Theorem 4.4 of McNeil (2008) provides sufficient conditions on the generator functions to guarantee that C is a copula. It holds that if $\phi_i \in \mathfrak{L}$ for $i = 1, \dots, d-1$ and $\phi_i \circ \phi_{i+1}$ has a completely monotone derivative for $i = 1, \dots, d-2$ then C is a copula.

Note that generators ϕ_i within a HAC can come either from a single generator family or from different generator families. If ϕ_i's belong to the same family, then the complete monotonicity of $\phi_i \circ \phi_{i+1}$ imposes some constraints on the parameters $\theta_1, \dots, \theta_{d-1}$. For the majority of the generators a feasible HAC requires decreasing parameters from the lowest to the highest level. However, in the case of different families within a single HAC, the condition of complete monotonicity is not always fulfilled.

In general the structure of the HAC can be arbitrary. This makes it a very flexible and simultaneously parsimonious distribution model. If we use the same single-parameter generator function on each level, but with a different value of θ, we specify the whole distribution with $d-1$ parameters. In contrary modeling with the Gaussian model puts the strong restriction of ellipticity on the dependency structure and requires $d(d-1)/2$ correlation coefficients. From this point of view, the HAC approach can be seen as a much more flexible alternative to Gaussian models.

2.1 Determination of the structure

In contrary to the classical parametric distributions, the estimation of the HAC is less straightforward. For each HAC not only the parameters are unknown, but also the structure has to be determined. One possible procedure is to enumerate and to estimate all possible HACs. Using an suitable goodness-of-fit test we can determine the optimal structure. This approach is unrealistic in practice even in moderate dimensions. Okhrin et al. (2009a) suggest a computationally efficient procedure, which allows to estimate the HAC recursively. Here we shortly sketch their idea and introduce the notation which is used throughout the paper.

Assume that each function ϕ_j depends on a single parameters θ_j. Let $X_1, \dots, X_d$ denote the set of random variables whose dependency structure we want to determine and let $(x_{1i}, \dots, x_{di})^\top$ be the respective samples for $i = 1, \dots, n$. For notational convenience let the expression $s = \{(\dots(i_1 \dots i_{j_1})\dots(\dots)\dots)\}$ denote the structure of the HAC, where $i_\ell \in \{1, \dots, d\}$ is a reordering of the indices of the variables. s_j denotes the structure of subcopulas with $s_d = s$. We assume that the variables X_i for $i = 1, \dots, d$ follow arbitrary continuous marginal distributions. Further let the d-dimensional hierarchical Archimedean copula be denoted by $C(\boldsymbol{\phi}, \boldsymbol{\theta}; s)(u_1, \dots, u_d)$, where $\boldsymbol{\phi}$ denotes the set of generating functions for all levels and $\boldsymbol{\theta}$ the set of copula parameters. As we consider the generator functions to be equal on all levels of the hierarchy we can avoid $\boldsymbol{\phi}$ in the argument list. For example the fully nested HAC can be expressed as

$$
\begin{aligned}
& C(\boldsymbol{\theta}; s = s_d)(u_1, \dots, u_d) = \\
= \quad & C(\{\theta_1, \dots, \theta_{d-1}\}; \{(s_{d-1})d\})(u_1, \dots, u_d) \\
= \quad & \phi_{d-1,\theta_{d-1}}(\phi^{-1}_{d-1,\theta_{d-1}} \circ C(\{\theta_1, \dots, \theta_{d-2}\}; ((s_{d-2})(d-1)))(u_1, \dots, u_{d-1}) \\
+ \quad & \phi^{-1}_{d-1,\theta_{d-1}}(u_d)),
\end{aligned}
$$

where $s = \{(\dots(12)3)\dots)d)\}$.

First we constrain the discussion to binary copulas, i.e. at each level of the hierarchy only two variables are joined together. The estimation of the copula is recursive. At the lowest level we fit a bivariate copula to every couple of the variables. The estimation procedure is discussed below. Using the results we select the couple of the variables with strongest fit and denote the respective estimator of the optimal parameter at the first level by $\hat{\theta}_1$ and the indices of the variable by I_1. The selected couple is joined together to define the pseudo-variables $C\{\phi_1, \hat{\theta}_1; (I_1)\}$. At the next level we proceed in the same way by considering the remaining variables and the new pseudo-variable. The considered procedure allows us to determine the optimal structure of the copula.

The estimation and the related asymptotic theory should be adopted to take into account the recursive procedure and the presence of pseudo-variables which depend on parameters, estimated at lower levels. The multi-stage maximum-likelihood estimation is a convenient tool in this case.

At the first stage we estimate the parameters of the marginal distributions in case of parametric margins or the nonparametric estimator of the marginal distribution in the nonparametric case. At the next stage we estimate the parameter of the copula at the first level assuming that the marginal distributions are known. At further stages the next level copula parameter is estimated assuming that the margins as well as the copula parameters at lower levels are known.

Assuming that d variables are joined within p hierarchical levels, which means that if $s = ((123)(45))$, then $s = s_3 = ((s_1)(s_2))$ where s_1, s_2, s_3 correspond to the levels 1, 2 and 3 respectively. Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)^\top$ be the parameters of copulas starting with the lowest up to the highest level. The multistage ML estimator $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\theta}^\top)^\top$ solves the system

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\alpha}_1^\top}, \dots, \frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\alpha}_k^\top}, \frac{\partial \mathcal{L}_{k+1}}{\partial \theta_1}, \dots, \frac{\partial \mathcal{L}_{k+p}}{\partial \theta_p}\right)^\top = \mathbf{0}, \tag{3}$$

$$\begin{aligned}
\text{where} \qquad & \mathcal{L}_j = \sum_{i=1}^{n} l_j(\mathbf{X}_i), \text{ for } j = 1, \dots, k+p, \\
& l_j(\mathbf{X}_i) = log\, f_j(x_{ji}, \boldsymbol{\alpha}_j), \text{ for } j = 1, \dots, k, i = 1, \dots, n, \\
& l_{j+k}(\mathbf{X}_i) = log\left[c(\{\theta_\ell\}_{\ell=1,\dots,p};\, s_j)\big(\{F_m(x_{mi}, \boldsymbol{\alpha}_m)\}_{m \in s_j}\big) \prod_{m \in s_j} f_m(x_{mi}, \boldsymbol{\alpha}_m)\right] \\
& \text{for } j = 1, \dots, p,\ i = 1, \dots, n.
\end{aligned}$$

where $\hat{F}_i(\cdot)$ is an estimator of the marginal CDF F_i. If we estimate the margins parametrically then $\hat{F}_i(\cdot) = F_i(\cdot, \hat{\boldsymbol{\alpha}}_i)$. $\hat{f}(\cdot)$ is the marginal density function estimated nonparametrically or parametrically in a similar way as the cumulative distribution function. Okhrin et al. (2009a) provide asymptotic behavior of the estimates.

Up to now we assumed that at each level of the hierarchy only two variables are joined together. If we can join more than two variables at once, we face two problems. First, a copula must be fitted at each level to each subset of the variables and not only to bivariate sets. This dramatically increases the number of distributions which must be estimated. Second, note that the copula density can be difficult to determine explicitly in high dimensions. Therefore, the ML estimation would require in practice numerical derivatives. This reduces the precision and increases the computational efforts. This argument also makes the reestimation of the copula with fixed structure computationally unattractive. As a solution Okhrin et al. (2009a) suggest to construct aggregated HAC on the basis of binary HACs relying in the proximity of the parameters.

3 Modeling of the Inhomogeneous Dependence

There numerous evidences in the literature that the correlation matrix is not constant over time. To model the time-varying correlation structure numerous models were proposed, with multivariate GARCH-type models among the most popular. In these models the correlations are defined as functions of (lagged) explanatory variables which may influence the variation in the correlation matrix. This implies that the conditional correlation changes at each moment of time. However, the parameters of the conditioning functions are assumed to be constant. There is a strong evidence that the parameters do change with time, see Lamoureux and Lastrapes (1990). Neglecting the variation in the parameters leads to inconsistent estimators and wrong inferences from the model. Models with time-varying parameters suffer, however, from the curse of dimensionality.

In this section we introduce a methodology which allows us to model the time-varying dependencies. In contrary to the conditional correlation models, we assume that there are periods of time where the dependency structure defined through a HAC is constant. This is a very parsimonious alternative to models with time-varying parameters. To determine the periods of homogeneity we rely on the local parametric fitting and local change point procedure developed and popularized by Polzehl and Spokoiny (2006) and Mercurio and Spokoiny (2004). The method can be virtually applied to any dependency model. However, applied to HACs, it allows us to control not only for the periods with constant parameters, but also for the periods with constant structure. Moreover, the method is efficient not only for abrupt changes in the dependency, but also for smooth transitions in the model parameters. This observation is particularly important in applications.

The moving window estimation of the distribution reflects the changes in the dependencies, but the length of the estimation window is mostly exogenously fixed. Let the time varying joint distribution F_t be given by $F_t(X_1, \ldots, X_d) = C_t\{F_{t,1}(X_1), \ldots, F_{t,d}(X_d); s_t, \boldsymbol{\theta}_t\}$. The use of the moving window of fixed width will estimate $\boldsymbol{\theta}_t$ and the structure s_t for each t. However, if the time elapsed since the last shift in the model heavily deviates form the window width, then the estimation results are misleading. In order to choose optimally the interval of homogeneity we use a local parametric fitting approach as introduced by Mercurio and Spokoiny (2004) and Härdle, Herwartz and Spokoiny (2002).

The basic idea is the adaptive estimate of the interval of homogeneity in which the hypothesis of a locally constant copula is supported. Using Local Change Point (LCP) detection procedure, see Spokoiny (2009), we sequentially test whether $\boldsymbol{\theta}_t$ is constant ($\boldsymbol{\theta}_t = \boldsymbol{\theta}$) and the structure of the HAC s_t is constant, i.e. $s_t = s$ within some interval I. We define the "oracle" choice as the largest interval $I = [t - m_{k^*}; t]$, for which the small modelling bias condition (SMB) is fulfilled

$$\triangle_I(s, \boldsymbol{\theta}) = \sum_{t \in I} \mathcal{K}(C_{s,\boldsymbol{\theta}}, C_{s_t,\boldsymbol{\theta}_t}) \leq \triangle \tag{4}$$

where $\boldsymbol{\theta}$ and s are fixed. $\mathcal{K}$ is the Kullback-Leibler divergence between the two copula-based CDFs. It measures the proximity between the two models. We discuss and justify it in more details in the next subsection.

The value m_{k^*} shows the largest number of recent observations which cover the period of constant dependency structure and must be used for estimation purposes. The error and risk bounds are calculated in Spokoiny (2009). Note that the aim of procedure is not to detect the shift, but to determine the period of constant dependency.

3.1 Kullback-Leibler Divergence

The Kullback-Leibler is used to quantify the difference between alternative CDFs. It belongs to the family of Ali-Silvey measures, see Basseville (1989).

Let $(X_1, \ldots, X_d)^\top \sim C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d); s, \boldsymbol{\theta}\}$. Then from Sklar (1959) the density function $f(\cdot)$ of the d-variate distribution F in terms of copula is given by

$$f(x_1, \ldots, x_d; s, \boldsymbol{\theta}) = c\{F_1(x_1), \ldots, F_d(x_d); s, \boldsymbol{\theta}\} \prod_{i=1}^{d} f_i(x_i), \quad x_1, \ldots, x_d \in \overline{\mathbb{R}},$$

where $c(\cdot; s, \boldsymbol{\theta})$ is the corresponding copula density defined as

$$c(u_1, \dots, u_d; s, \boldsymbol{\theta}) = \frac{\partial^d C(u_1, \dots, u_d; s, \boldsymbol{\theta})}{\partial u_1 \dots \partial u_d}. \tag{5}$$

The Kullback-Leibler divergence for copulae is developed to measure the distance between two models. From the definition it follows, that

$$\begin{aligned}
\mathcal{K}\{F(\cdot; s_0, \boldsymbol{\theta}_0), F(\cdot; s_1, \boldsymbol{\theta}_1)\} &= E_{s_0, \boldsymbol{\theta}_0} \log \frac{f(x_1, \dots, x_d; s_0, \boldsymbol{\theta}_0)}{f(x_1, \dots, x_d; s_1, \boldsymbol{\theta}_1)} \\
&= E_{s_0, \boldsymbol{\theta}_0} \log \frac{c\{F_{X_1}(x_1), \dots, F_{X_d}(x_d); s_0, \boldsymbol{\theta}_0\}}{c\{F_{X_1}(x_1), \dots, F_{X_d}(x_d); s_1, \boldsymbol{\theta}_1\}}
\end{aligned}$$

In the empirical and simulation part the expectation is estimated through the average over the simulation runs

$$\widehat{\mathcal{K}}(F_1, F_2) = \frac{1}{n} \sum_{i=1}^{n} \log \frac{f_1(x_{1i}, \dots, x_{di})}{f_2(x_{1i}, \dots, x_{di})},$$

where the $\{x_{1i}, \dots, x_{di}\}_{i=1}^{n}$ is the sample of size n from a d-dimensional distribution.

It is easy to see, that Kullback-Leibler measures are non symmetric and that for independence copula given by $C_{\perp}(u_1, \dots, u_d) = u_1 \dots u_d$ holds

$$\begin{aligned}
\mathcal{K}(C_{\perp}, C_{s,\boldsymbol{\theta}}) &= -E_{\perp}\{\log c(u_1, \dots, u_d; s, \boldsymbol{\theta})\} \\
\mathcal{K}(C_{s,\boldsymbol{\theta}}, C_{\perp}) &= E_{\perp}\{\log c(u_1, \dots, u_d; s, \boldsymbol{\theta})\}
\end{aligned}$$

Particular role plays the Kullback-Leibler divergence in the analysis and estimation of misspecified models (see White (1982)). Let s_0 be the correctly specified structure with the parameters $\boldsymbol{\theta}_0$ and s_1 be the misspecified but fixed structure. Note that s_0 and $\boldsymbol{\theta}_0$ are usually unknown. Then minimizing the KL-divergence with respect to $\boldsymbol{\theta}_1$ is equivalent to ML estimation of the HAC with misspecified structure. In this case we minimize our ignorance of the true model. If s_0 and s_1 coincide, then the estimator of $\boldsymbol{\theta}_1$ is also a consistent estimator of the true parameter $\boldsymbol{\theta}_0$. Note that

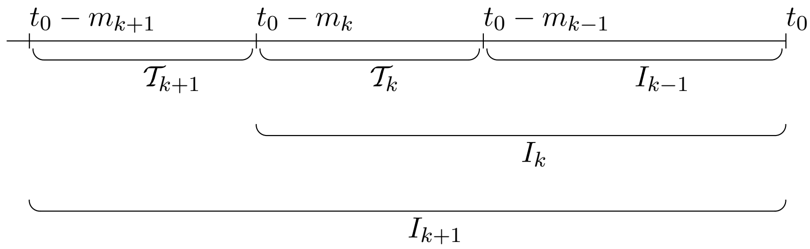
$$\mathcal{K}\{F(\cdot; s_0, \boldsymbol{\theta}_0), F(\cdot; s_1, \boldsymbol{\theta}_1)\} = E_{s_0, \boldsymbol{\theta}_0} \log f(x_1, \dots, x_d; s_0, \boldsymbol{\theta}_0) - E_{s_0, \boldsymbol{\theta}_0} \log f(x_1, \dots, x_d; s_1, \boldsymbol{\theta}_1).$$

The first term is independent on s_1 and $\boldsymbol{\theta}_1$, while the second term can be estimated by the sample log-likelihood function. In the case of minimizing $\triangle_I(s, \boldsymbol{\theta}) = \sum_{t \in I} \mathcal{K}(C_{s,\boldsymbol{\theta}}, C_{s_t, \boldsymbol{\theta}_t})$ with respect to m_k, where s_t and $\boldsymbol{\theta}_t$ are true but unknown time-varying copula parameters, we minimize our ignorance of the time variation in the copula. In the next subsection we present an adaptive procedure which estimates the copula with the same precision as the oracle with true but unknown s_t and $\boldsymbol{\theta}_t$.

3.2 Local Change Point Procedure

The homogeneity interval is selected by the local change point (LCP) detection procedure that is based on the adaptive choice of the interval of homogeneity for the endpoint t_0. Let $I = I_k, k = -1, 0, 1, \dots$ be a family of intervals such that $I_k = [t_0 - m_k, t_0]$ with $\{m_k\}$'s defined such

Figure 1: LCP Procedure



that $m_{-1} < m_0 < \ldots \leq t_0,\ m - 1 = \rho_2 m_1,\ m_0 = \rho_1 m_1$ and $\rho_1 > \rho_2 \in (0, 1)$. Additionally let $\mathcal{T}_k \subset I_k$ be the sets of internal points of the form $\mathcal{T}_k = [t_0 - m_{k-1}, t_0 - m_{k-2}]$ for $k = 1, 2, \ldots$.

We start the procedure with $k = 1$ and (see Figure 3.2) test the hypothesis $H_{0,k}$ of homogeneity of the copula against the change point alternative in $\mathcal{T}_k$ using the observations in I_{k+1}. If there are no change points in $\mathcal{T}_k$ then we accept I_k as an interval with constant copula parameters. At the next step we take $\mathcal{T}_{k+1}$ and repeat the previous steps until $H_{0,k}$ is rejected or the largest possible interval I_K is accepted. In the case if $H_{0,k}$ is rejected in $\mathcal{T}_k$, then the homogeneity interval is the last accepted $\hat{I} = I_{k-1}$. The largest possible interval that can be accepted is $\hat{I} = I_K$. The local homogeneity test can be performed in the following way. We estimate the copula parameter $\boldsymbol{\theta}$ and the structure s from observations in $\hat{I}$, assuming the homogeneous model within $\hat{I}$, i.e. we define $\hat{\boldsymbol{\theta}}_{t_0} = \tilde{\boldsymbol{\theta}}_I$ and $\hat{s}_{t_0} = \tilde{s}_I$. Let $I = [t_0 - m, t_0]$ be an interval candidate and $\mathcal{T}_I$ be a set of interval points within I. The null hypothesis H_0 means that $\forall \tau \in \mathcal{T}_I:\ \boldsymbol{\theta}_t = \boldsymbol{\theta},\ s_r = s$ i.e. the observations in I follow the model with the dependence parameter $\boldsymbol{\theta}$ and the structure s. The alternative hypothesis H_1 claims that $\exists \tau \in \mathcal{T}_I : \boldsymbol{\theta}_t = \boldsymbol{\theta}_1$ and $s_t = s_1$ for $t \in J = [\tau, t_0]$ and $\boldsymbol{\theta}_t = \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_1$ or $s_t = s_2 \neq s_1$ for $t \in J^c = [t_0 - m, \tau)$, i.e. either the parameter $\boldsymbol{\theta}$ or the whole structure s changed spontaneously at some intermediate point τ of the interval I. In other words

$$
\begin{aligned}
H_0 &: \ \forall \tau \in \mathcal{T},\ \theta_t = \boldsymbol{\theta},\ s_t = s,\ \forall t \in J = [\tau, t_0], \forall t \in J^c = I \setminus J \\
H_1 &: \ \exists \tau \in \mathcal{T},\ \theta_t = \theta_1,\ s_t = s_1;\ \forall t \in J,\ \theta_t = \theta_2 \neq \theta_1 \text{ or } \ s_t = s_2 \neq s_1, \forall J^c.
\end{aligned}
$$

If $\mathcal{L}_I(s, \boldsymbol{\theta})$ and $\mathcal{L}_J(s_1, \boldsymbol{\theta}_1) + \mathcal{L}_{J^c}(s_2, \boldsymbol{\theta}_2)$ are the log-likelihood functions corresponding to H_0 and H_1 respectively, the likelihood ratio test for the single change point with known fixed location τ is given by

$$
\begin{aligned}
T_{I,\tau} &= \max_{s_1, \boldsymbol{\theta}_1, s_2, \boldsymbol{\theta}_2} \{\mathcal{L}_J(s_1, \boldsymbol{\theta}_1) + \mathcal{L}_{J^c}(s_2, \boldsymbol{\theta}_2)\} - \max_{s, \boldsymbol{\theta}} L_I(s, \boldsymbol{\theta}) \\
&= ML_J + ML_{J^c} - ML_I.
\end{aligned}
$$

Since the time point of the change is unknown, we introduced a test statistics with is independent on τ, i.e.

$$
T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}
$$

It tests the homogeneity hypothesis in I against the change point alternative with unknown location τ, which belongs to the set of considered locations $\mathcal{T}_I$. The decision rule of the test requires to compare the test statistics with the critical value λ_I. The critical value may depends on the interval I, dimension and parameters of the copula. We reject the hypothesis of homogeneity if $T_I > \lambda_I$. The estimator of the change point is then defined as

$$\hat{\tau} = \arg\max_{\tau \in \mathcal{T}_I} T_{I,\tau}.$$

To run the test several parameters have to be specified. This includes the choice of the interval candidates I and internal points $\mathcal{T}_I$ for each of this intervals and the choice of the critical values λ_I. One possible example of the implementation is based on the choice of the interval candidates I in form of a geometric grid. If the length of the interval I_1 if fixed at m_1, then we define $m_0 = \rho_1 m_1$ and $m_{-1} = \rho_2 m_1$ for $\rho_1 > \rho_2 \in (0,1)$ and $m_k = [m_1 c^{k-1}]$ for $k = 1, 2, \ldots, K$ and $c > 1$, where $[x]$ means the integer part of x. Further we set $I_k = [t_0 - m_k, t_0]$ and $\mathcal{T}_k = [t_0 - m_{k-1}, t_0 - m_{k-2}]$ for $k = 1, 2, \ldots, L$. Clearly there is no explicit expression for the critical value λ_I. For each particular true copula model and for each length of the interval it should be determined from simulations.

4 Simulation Study

In this subsection we apply the LCP procedure to the data simulated from a HAC. The purpose of the study is to assess how precisely the estimation procedure can follow the shift in the parameters and/or in the structure. We consider a 3-dimensional HAC with Gumbel generators and simulate 100 samples of length 1000. Margins are uniformly distributed and to simulation from a HAC we used the algorithm of McNeil (2008). The shift occurs at the observation 500. The change point in the model is modeled in two different ways. In the case of the shift at the lower level of the copula, it is modeled as

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 1.43); \theta_2 = 1.11\} & \text{for } 1 \le t \le 500 \\ C\{u_1, C(u_2, u_3; \theta_1 = 1.25); \theta_2 = 1.11\} & \text{for } 500 < t \le 1000 \end{cases}, \tag{6}$$

while the shift at the upper level we model by

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 1.43); \theta_2 = 1.11\} & \text{for } 1 \le t \le 500 \\ C\{u_1, C(u_2, u_3; \theta_1 = 1.43); \theta_2 = 1.25\} & \text{for } 500 < t \le 1000 \end{cases}. \tag{7}$$

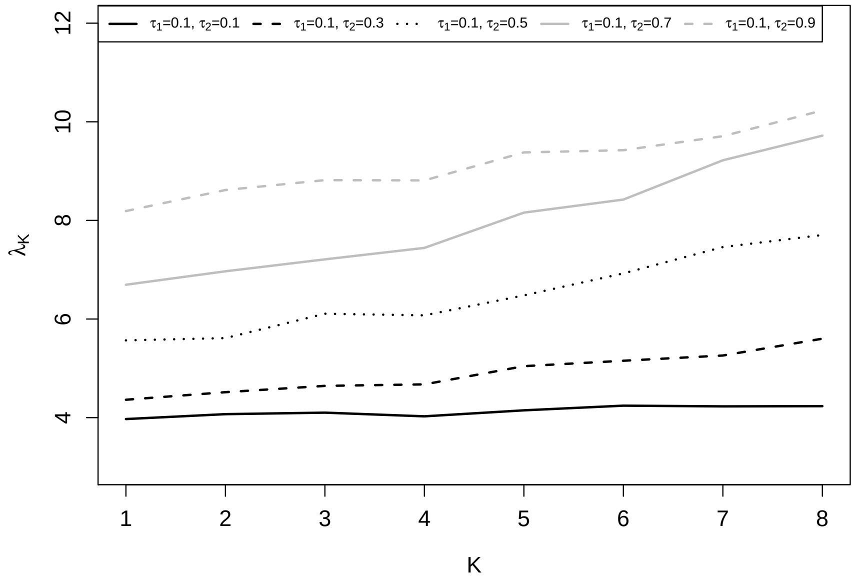
The initial parameters $\theta_1 = 1.43$ and $\theta_2 = 1.11$ correspond to the Kendall τ's equal to 0.3 and 0.1 respectively. After the shift of the first type τ_1 decreases to 0.2, while for the second type of the shift τ_2 increases to 0.2. Note that in both cases the difference between the parameters becomes smaller. This requires the procedure to be more sensitive to changes, in contrary to the shifts, which lead to an increase in the difference.

The change point in the structure is modeled in a similar way

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 1.25); \theta_2 = 1.11\} & \text{for } 1 \le t \le 500 \\ C\{C(u_1, u_2; \theta_1 = 1.25), u_3, ; \theta_2 = 1.11\} & \text{for } 500 < t \le 1000 \end{cases}. \tag{8}$$

The LCP procedure is implemented with the family of interval candidates in the form of geometric grid defined by $m_0 = 20, 40$ and $c = 1.25$ as suggested Giacomini et al. (2009). The

Figure 2: Critical values of the 3-dimensional HAC for different K, θ_1, θ_2 and $m_0 = 20$



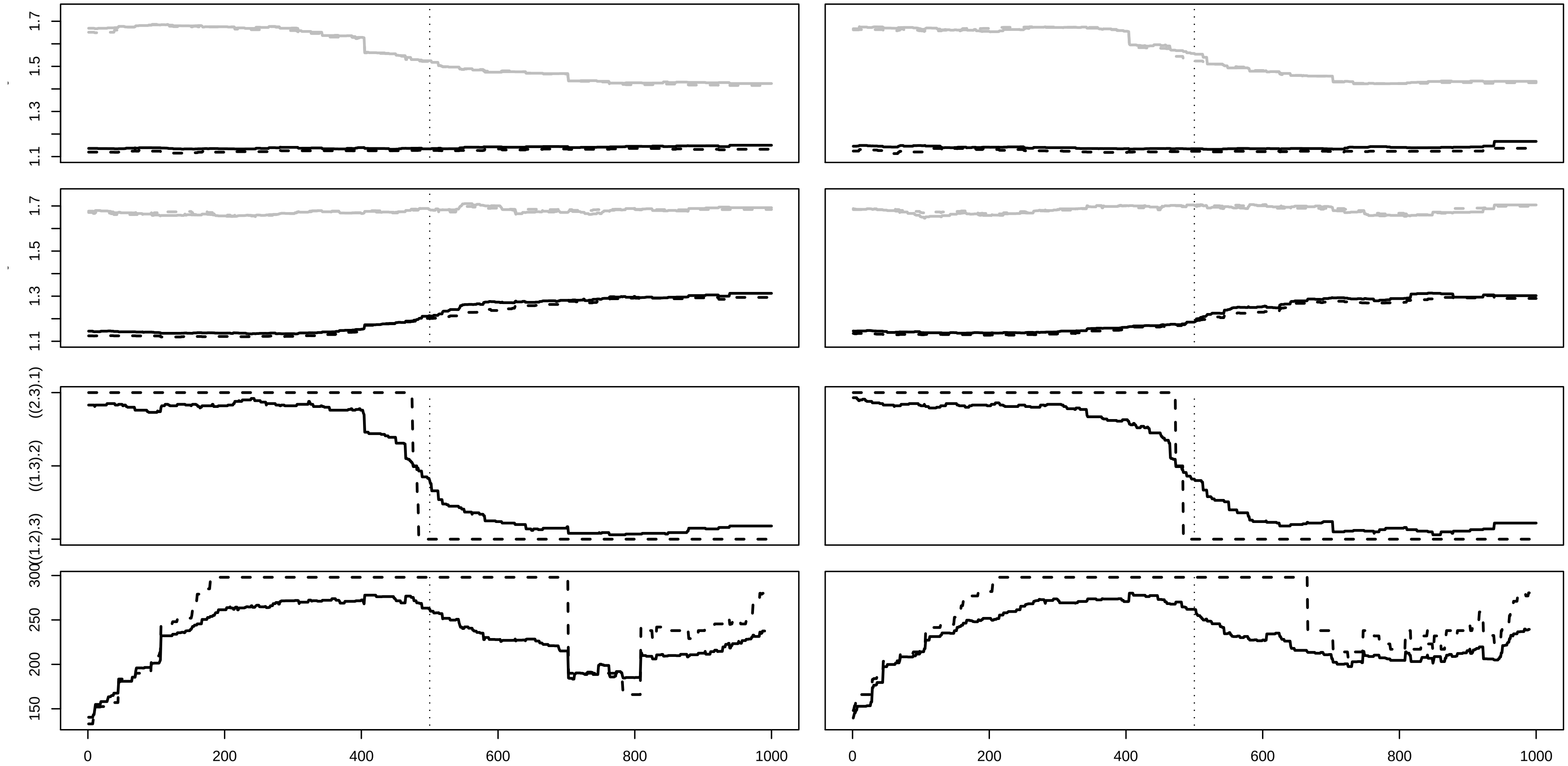
decision rule additionally depends on the critical value, which is determined numerically in a simulation study. Note that the critical value is indifferent to the form of the initial structure $s_1 = ((12)3)$ or $s_2 = (1(23))$, but depends on the parameters. Using the fact that for the Gumbel copula the parameter $\theta \in [1;\infty)$ is unbounded from above, we define the grid based on the Kendall's τ by

$$\boldsymbol{\theta} = \{\theta_1, \theta_2\} = \{\theta(\tau_1), \theta(\tau_2)\},$$

where

$$\{\tau_1, \tau_2\} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}^2, \ \tau_1 \geq \tau_2.$$

Figure 3: The average and median estimated parameters and structure on the intervals of homogeneity.



The estimated parameters and structure on the intervals of homogeneity. The solid line depicts the average structure/parameters and the dashed line the median structure/parameters over 100 replications. The shift occurs at the observation 500. m_0 is set to 20 for the left panel and to 40 for the right panel. The first pair illustrate the change-point model in (6), the second pair the model in (7) and the third in (8).

This grid on τ corresponds to the grid for θ's given by $\{\theta_1, \theta_2\} \in \{1.11, 1.43, 2, 3.33, 10\}^2$, $\theta_1 \geq \theta_2$. Thus we simulate from copulas $C\{u_1, C(u_2, u_3; \theta_1 = 1.43); \theta_2 = 1.11\}$, $C\{u_1, C(u_2, u_3; \theta_1 = 2.00); \theta_2 = 1.11\}$, etc. The case $\theta_1 = \theta_2$ corresponds to the simple 3-dimensional Archimedean copula $C(u_1, u_2, u_3, \theta_1)$. To estimate λ we simulate $n = 5000$ samples of size $k = 1000$. Using the same geometric grid of the intervals we calculate the test statistics $T_i(K, \theta_1, \theta_2)$, $i = 1, \ldots, n$, $K = 1, \ldots, 8$ and θ_1, θ_2. Finally the critical value is defined as the quantile of the distribution of $T_i(K, \theta_1, \theta_2)$

$$\lambda(K, \theta_1, \theta_2) = \tilde{T}_{0.95}(K, \theta_1, \theta_2)$$

where $\tilde{T}_{0.95}$ denotes the 95% -quantile of the sample. Figure 2 shows the behavior of the critical values as a function K for different values of τ_2.

In the simulation study we set $k = 1000$. We start the estimation procedure at the end of the sample. Applying LCP to the recent observations we determine the interval with constant dependency and estimate the corresponding HAC. At the next step we fix the time point preceding the last homogeneity interval and apply the LCP to further historical observations. The results are shown in Figure 3. m_0 is set to 20 in the left column and to 40 in the rights column. The solid lines show the average values and the dashed line the median values over 100 replications. The first two figures illustrate the application of LCP to the change-point model (6), while the second type of the shift (7) is plotted in the second row. The change in the structure is illustrated in the third row, where we coded each of three structures for convenience with 1,2 and 3.

For all three types of the shift, we observe that the average estimated parameter or structure smoothly moves from the value before the shift to the value after the shift. The speed of transition strongly depends on m_0. Smaller values of m_0 imply that we can use less observations to estimate the distribution. This allows the procedure to react quickly to any changes. On the other hand the precision of the estimation decreases with decreasing sample size. The last two figures in Figure 3 show the average length of the interval of homogeneity for given time point. We start with the shortest available interval of homogeneity. Since the copula is stable and more observations become available, the length of the interval increases to the largest allowed value. After the shift the length of the interval decreases and increases only after the change-point leaves the smallest allowed interval.

5 Empirical Study

In the last section we observed that the LCP is very successful in modeling changing dependencies. Here we apply it to real data. In the empirical study and to calibrate the setup of the simulation study we use the daily log-returns X_t for four companies from the DAX index: Commerzbank (CBK), Merck KGAA (MRK), Thyssenkrupp (TKA) and Volkswagen (VOW) for the period [13.11.1998; 18.10.2007], which covers more than 2300 trading days. The data is obtained from `DataStream`.

To eliminate the conditional heteroscedasticity we fit to each time series of log-returns a univariate GARCH(1,1) process

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t} \text{ with } \sigma^2_{j,t} = \omega_j + \alpha_j \sigma^2_{j,t-1} + \beta_j (X_{j,t-1} - \mu_{j,t-1})^2$$

Table 1: Estimation results univariate time series modelling.

	$\hat{\mu}_j$	$\hat{\omega}_j$	$\hat{\alpha}_j$	$\hat{\beta}_j$	BL	KS
CBK	5.200e-04	8.600e-06	9.751e-02	8.864e-01	0.04109	1.224e-05
	(3.210e-04)	(1.866e-06)	(1.202e-02)	(1.341e-02)		
MRK	7.392e-04	4.588e-06	3.333e-02	9.572e-01	0.1285	1.255e-11
	(3.672e-04)	(1.557e-06)	(6.225e-03)	(8.568e-03)		
TKA	7.845e-04	3.549e-06	7.087e-02	9.252e-01	0.1360	4.189e-05
	(3.308e-04)	(1.149e-06)	(9.837e-03)	(9.915e-03)		
VOW	9.720e-04	1.239e-05	9.303e-02	8.830e-01	1.927e-05	3.422e-06
	(3.480e-04)	(2.699e-06)	(1.301e-02)	(1.566e-02)		

Note: Results of the fitting of univariate GARCH(1,1) to asset returns. The standard deviation of the parameters are given in parentheses. The last two columns provide the p-values of the Box-Ljung test (BL) for autocorrelations and Kolmogorov-Smirnov test (KS) for normality applied to the residuals.

and $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $\alpha_j + \beta_j < 1$. The estimates of the parameters are given in Table 1. The parameters of the equation for conditional volatility are strongly significant for all time series. The table also contains the p-values of the Box-Ljung test with 12 lags and Kolmogorov-Smirnov test applied to the residuals. We conclude that the residuals exhibit a typical behavior for financial time series. The residuals are not normally distributed, which is an evidence in favor of the nonparametric estimation of the margins. From the results of the Box-Ljung test we conclude that the autocorrelation of the residuals is strongly significant only for VOW. In the rest of the paper we work only with the residuals.

If we consider multivariate observations, then monitoring and measuring changes in the overall dependency is not a straightforward problem. To motivate the application of LCP procedure we first asses the variation of dependency measured by Kendall and Pearson's correlation coefficients. Figure 4 shows the behavior of the Pearson correlation coefficients estimated using rolling window of width $r = 250$ observations. The correlation coefficients show clear variation in time with clear upward and downward trends. The behavior of Kendall and Pearson correlation is similar, but not identical. This implies that a copula-based model may be successful in capturing the dependencies in the data.

To give an additional justification for the use of a copula-based distribution to model the residuals we compute the maximum-likelihood (ML) and the Bayes information criterion (BIC) for rolling window fit of alternative models to the data. We consider the binary HAC with Gumbel generator; the 4-dimensional Gaussian distribution and 4-dimensional t-distribution. The maximum-likelihood for the copula-based distribution is computed as

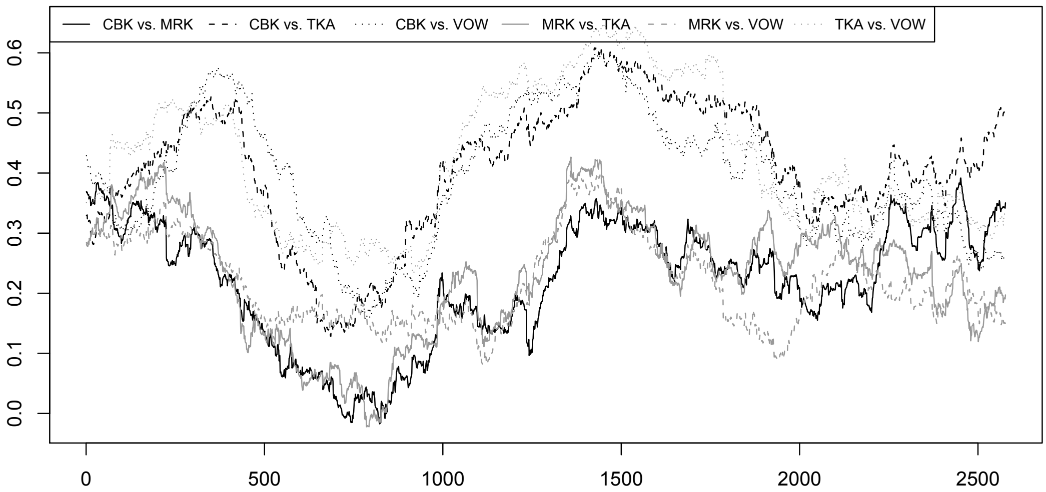
$$ML = \sum_{i=1}^{n} \log\{c(u_1, \dots, u_d; s, \boldsymbol{\theta}) f_1(u_1) \dots f_d(u_d)\},$$

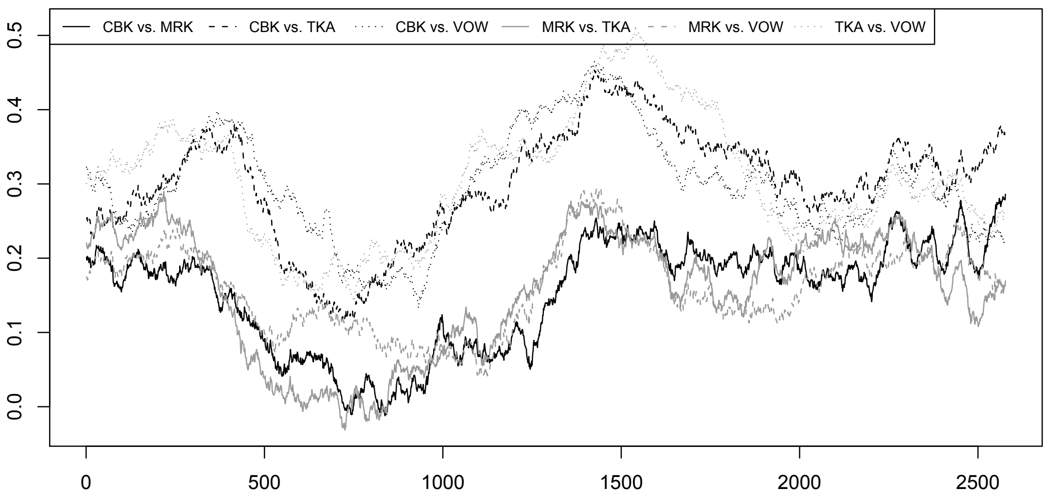
and for the multivariate normal and t-distributions as

$$ML = \sum_{i=1}^{n} \log\{f(u_1, \dots, u_k; \boldsymbol{\Sigma}, \boldsymbol{\mu}, df)\},$$

where f denotes the joint multivariate density and $\boldsymbol{\Sigma}$ is the covariance matrix, $\boldsymbol{\mu}$ is the mean and df is the number of degrees of freedom in case of t-distribution. The BIC criterion is computed by $BIC = -2ML + 2\log(m)$, where m is the number of the parameters to be estimated.

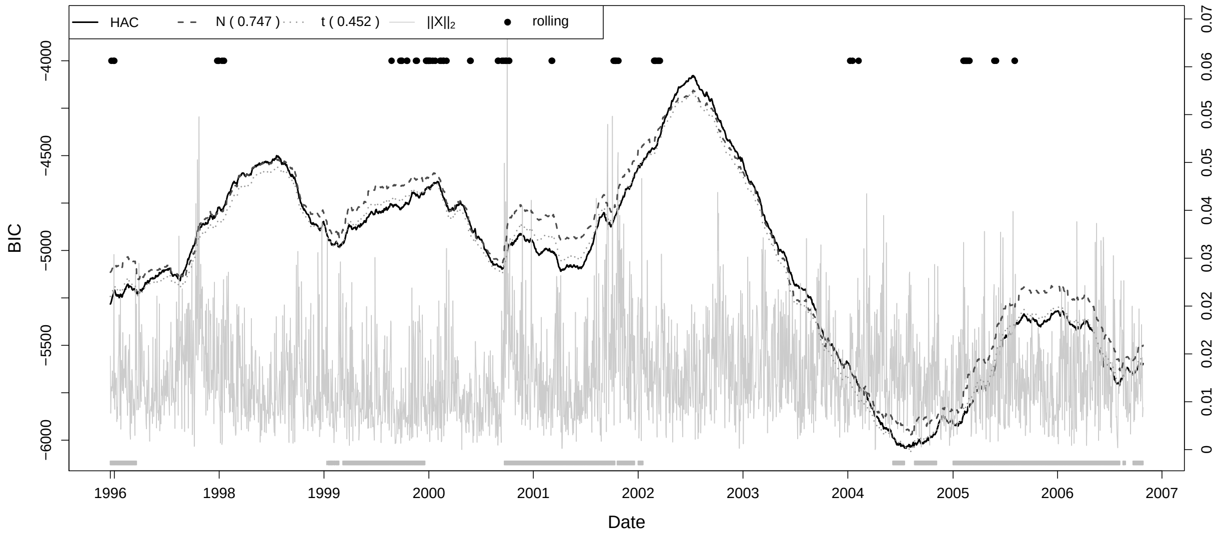
Figure 4: Rolling window estimators of Pearson's and Kendall's correlation coefficients.





Note: Rolling window estimators of Pearson's (top) and Kendall's (bottom) correlation coefficients or the residuals from GARCH(1,1) modelling of asset returns. The width of the rolling window is set to 250 observations.

Figure 5: Time-varying HAC: BIC



Note: Rolling window estimator of BIC for HAC-base, multivariate normal and t-distributions. The width of the rolling window is 250 observations. The grey line shows the variation of L_2 norm of the difference in the parameter matrices of the copulas. The dots mark the changes in the structure of binary copula using rolling window estimation.

As it is incorrect to compare the models with nonparametrically and parametrically estimated margins because the number of unknown parameters in the nonparametric case is unknown. Thus in those cases we consider only those parameters that are to be estimated for the copula function. Figure 5 illustrates the dynamics of BIC for four multivariate models. The values in the legend show the percentage times where the aggregated HAC outperforms the other models. The HAC with binary structure outperforms the normal and t distributions in 74.7% and 45.2% of the cases respectively.

Next we verify if the variation in the dependency can be linked to some characteristics of the distribution. The dots in Figure 5 show the time-points of changes in the bivariate HAC estimated using rolling window procedure. There is no visible relationship between the dynamics of the model fit measured by BIC and the changes in the structures. The thin grey shows the dynamics of the $||\widehat{\boldsymbol{\Theta}}_t - \widehat{\boldsymbol{\Theta}}_{t-1}||_2$, where $\widehat{\boldsymbol{\Theta}}_t$ denotes the matrix of copula parameters estimated at the time point t and $||\cdot||_2$ denotes the L_2 matrix norm. It is defined as $||\mathbf{A}||_2 = \sqrt{\lambda_{max}(\mathbf{A}^\top \mathbf{A})}$, where λ_{max} is the largest eigenvalue of the matrix $\mathbf{A}^\top \mathbf{A}$ (Perron root). Similarly as for BIC, there is no clear relationship between the changes in the structure and the variation in copula parameters. This implies that there is no obvious way how can we exploit the results from rolling window estimation to determine the intervals with homogeneous dependency.

6 Application of LCP

The previous section provided evidence on two important issues. First, the univariate marginal distributions are not normal and the joint distribution can be better modelled using a HAC-based distribution. Thus justifies the estimation of the joint distribution by HAC and nonparametrically estimated margins. Second, the dependency is not constant and varies with time. Since we model the dependency by HAC, this implies that either the structure of HAC or the copula parameters are time-dependent. In this section we apply the LCP procedure to compute a robust estimator of HAC.

6.1 3-dimensional Case

For simplicity we consider first a 3-dimensional case with three stocks CBK, MRK and TKA. As before the set of numbers m_k defining the length of I_k and $\mathcal{T}_k$ are in the form of a geometric grid with $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 = 20$ or 40 and $c = 1.25$, where $[x]$ denotes the integer part of x. The critical values λ are taken from the simulation study. The structure estimated from the whole data sample is given by $s^* = ((1.3)_{1.331}.2)_{1.145}$.

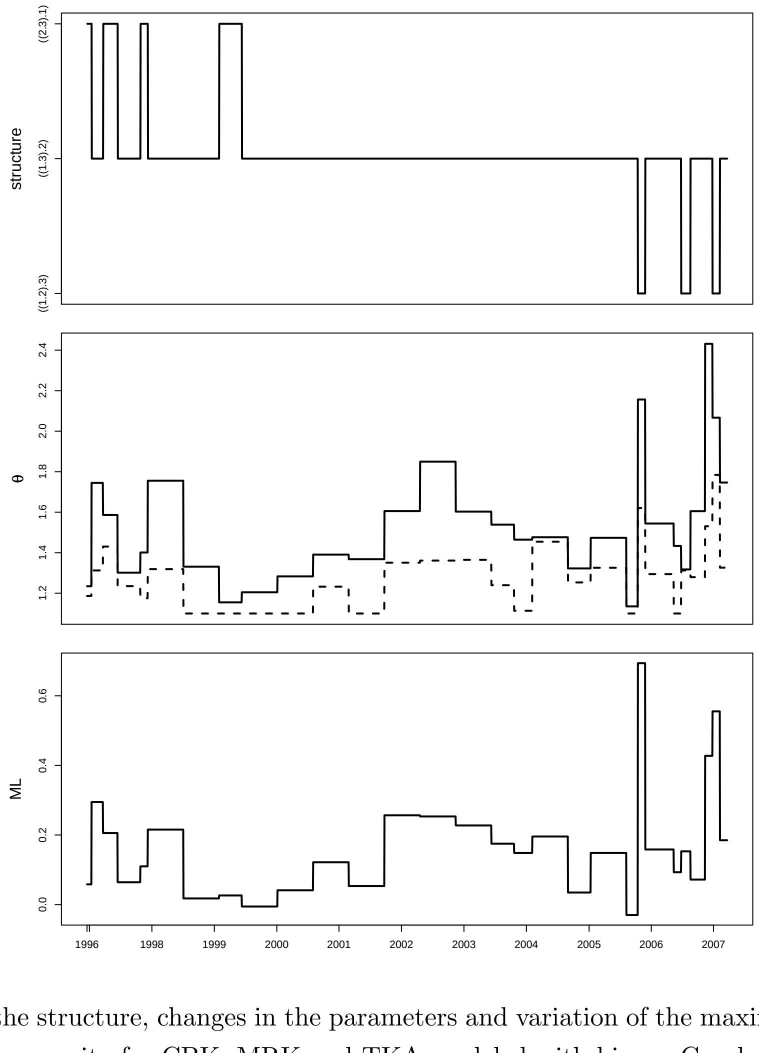
Figure 6 illustrated the results of the application of LCP. The upper figures shows the changes in the structure. There are very few shifts and the structure ((1.3).2) is very robust. This structure also coincides with the structure estimated from the whole sample. Nevertheless, the parameters do change. Their variation over the intervals over homogeneity is shown in the second figure. We see that the LCP captures even relatively small changes in parameters. Recall that the LCP procedure is based on the stability of the fit measured by maximum-likelihood. The values of ML criteria computed over the intervals of homogeneity are given at the bottom of Figure 6. The criteria shows strong variation even on the intervals with equal structure. This implies that the adaptive estimation procedure for the parameters partially fails to attain the same level of fit of the HAC. On the other hand, shift in the structure are always linked to strong changes in the parameters and in the fit of the model. This implies that the change in the distribution is substantial and cannot be neglected.

6.2 4-dimensional Case

Next we consider the complete data set consisting of 4 time series of returns for CBK, MRK, TKA and VOW. In the 3-dimensional case the computation of the critical values was less computationally demanding, because the critical values are determined by only two parameters. In the 4-dimensional case we have three parameters. This requires the computation of critical values on a 3-dimensional grid. To reduce the computational efforts we proceed in the following way. We estimate the HAC for the full data sample and use it the true structure. It is given by $s^* = ((1.4)_{1.40}.3)_{1.36}.2)_{1.11}$. Further the calculation of the critical values $\lambda(K)$ are based only on this structure and not on the grid as in the 3-dimensional example. This implies that the critical values depend only on K and not the parameters of copula estimated on the previous interval of homogeneity. The resulting $\lambda(K)$ are plotted in Figure 7.

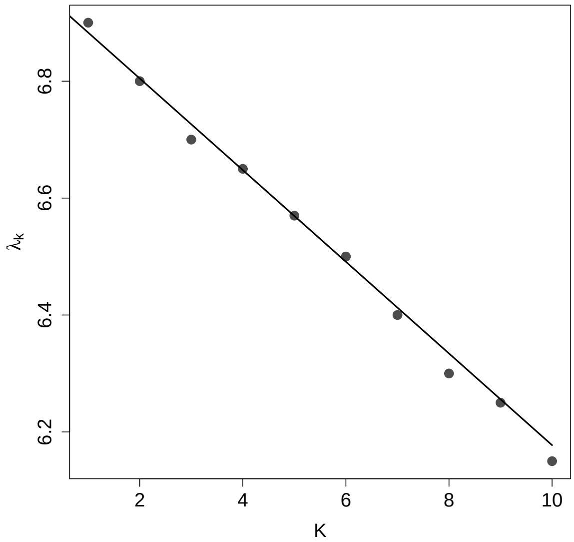
The lengths m_k defining the intervals I_k and $\mathcal{T}_k$ are chosen in the form of a geometric grid with $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 = 20$ and $c = 1.25$, where $[x]$ denotes the integer part of x. The estimation results are plotted in Figure 8. The top figure contains the variation of the estimated structures. Note that the structures cannot be naturally ordered. For illustration purposes we order the structures using the following procedure. We enumerate all possible structures and estimate the parameters for each of them from the full data sample. The structures are then ordered by the value of ML criterion. Since more structures are available

Figure 6: Structure, parameters and ML of the estimated HAC on the intervals of homogeneity, $m_0 = 20$



Note: Changes in the structure, changes in the parameters and variation of the maximum-likelihood over the intervals of homogeneity for CBK, MRK and TKA modeled with binary Gumbel HAC. m_0 is set to 20.

Figure 7: Critical values of the 4-dimensional HAC for different K and $s^* = ((1.4)_{1.40}.3)_{1.36}.2)_{1.11}$



to model a 4-dimensional data set, the variation in the structures is much stronger than in the 3-dimensional case. Nevertheless, the frequent structures are close the structure estimated from the whole sample. The same refers to the variation in the parameters. The parameters for the lower structures are close. The figure at the bottom shows the variation of the ML criteria over the intervals of homogeneity. The high structures exhibit lower values of ML. The LCP procedure allows us to find periods with such particular structures. By the estimation from the full sample the heterogeneity between the intervals is not observable. This provides an additional evidence in favor of application of the LCP procedure to monitoring the parameters and the structure of HACs.

7 Conclusions

In this paper we suggest a method of estimating the time-varying dependencies. The joint distribution of multivariate observations is modeled by a Hierarchical Archimedean copula. Using the Local Change Point detection procedure we determine the intervals with homogeneous dependency structure. In contrary to non-copula-based distributions, we succeed in modeling both homogeneity of the copula structure and homogeneity of the copula parameters. The procedure was evaluated in an extensive simulation study and compared to the classical rolling window estimation. Application to real data disclosed interesting features of the dynamics of dependencies.

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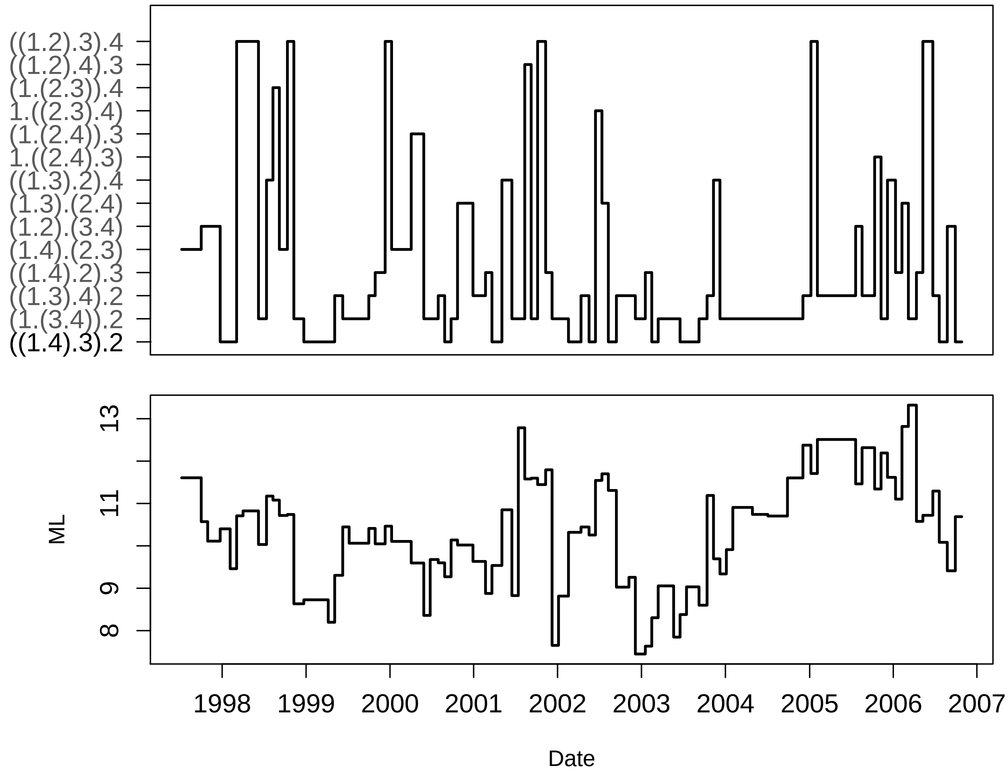
Figure 8: Local change point detection for the 4-dimensional HAC



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