

A Generalized Real Options Model of Land: Valuation, Expected Returns and Development

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Abstract

We develop an equilibrium model of land valuation, expected returns and development. The model extends the standard real options framework by combining cash flow uncertainty with uncertain buyer reservation prices. These reservation prices also serve as a proxy for general economic conditions. The model allows for the realism of lumpy uncertain demand and absorption of space over time where renters/buyers arrive and depart according to a Poisson process with a stochastic intensity that is a function of the uncertain variables. Buyers and renters can arrive and depart before construction has commenced, while additional buyers/renters can arrive during and after construction. This structure allows for presales/preleasing of space to mitigate developer's risk. Given various parameter sets, we numerically solve the model and derive numerous outputs including: the equilibrium value of undeveloped land, the equilibrium value of developed space, the optimal construction commencement trigger as a function of the presales, price and expected absorption levels, the expected time to construction, and the expected return on land investment.

1. Introduction

Researchers have long recognized holding vacant land as an option granting the right to develop a building in the future. Titman (1985), Capozza and Helsley (1990), Williams (1991), Grenadier (1995) and others have used this analogy to develop theoretical real-option models revealing the importance of uncertainty in the valuation of land and optimal timing of the development decision. Williams (1993) and Grenadier (2005) have extended the models employing a partial equilibrium framework by incorporating supply competition among developers and uncertain demand in a continuous-time Nash equilibrium. Williams (1993) shows how land values and development decisions are affected in the presence of less than perfect competition and downward sloping demand. Grenadier (2005) uses a similar setting to derive the stochastic properties of the equilibrium cash flow levels resulting in the derivation of the term structure of lease rates.

Several empirical studies have tested and validated theoretical real-option models for land valuation and development decision-making. Holland, Ott and Riddiough (2000) specify and empirically estimate a structural model of the commercial real estate asset market equilibrium. They find the real-options model in general—and the uncertainty variable in particular—has significant power in explaining aggregate commercial real estate investment and development cycles.¹ Other studies by Quigg (1993) and Cunningham (2006) provide empirical evidence of real option values embedded in land prices in the residential housing market.

¹ Other related empirical work includes a study by Sivitanidou and Sivitanides (2000). Examining investment over time for the U. S office market, they find results consistent with those of Holland, Ott and Riddiough (2000).

This paper builds upon previous theoretical research by employing a methodology similar to that of Williams (1993) and Grenadier (2005) to model the real estate market in equilibrium. We modify their structure of real estate markets and land development by relaxing the assumption that tenants or buyers are continuously available in the market. Uncertain demand and absorption of space are alternatively modeled as a lumpy process. Specifically, we treat demand as tenants/buyers arriving and departing the market according to a Poisson process. The stochastic intensity of this demand is a function of the ratio of the market service/cash flow to the mean buyer's reservation service/cash flow, both of which are stochastic.² The reservation flow also serves as a proxy for general economic conditions. This approach allows us to model lumpiness and uncertainty in space absorption, which lets us address real risks faced by developers, such as the risk of unsold inventory and/or declining prices after construction has begun. We provide for the ability of the developer to prelease or pre-sell space before, during and after construction to mitigate these risks.^{3,4} Optimal development decisions are shown to be a function of presale/prelease levels, the intensity function described above, and expected absorption levels.

Similar to Grenadier (2005), we find equilibrium building cash flows at the point where additional supply is added to the market. Equilibrium cash flow in our model is the

² If we are modeling the residential housing market we would use a service flow. For the commercial real estate markets we would use cash flow.

³ The basic option-based model has been extended to examine strategic real estate development. Grenadier (1996) shows that investment timing options for projects held by competitors, combined with the strategically optimal exercise of these options, can explain both simultaneous investment in competing projects and long lag times between investments. Grenadier (1999) and Childs, Ott and Riddiough (2002) extend this strategic setting, to one where there is either asymmetric or incomplete information among competing developers. In these models actual development reveals information to competitors. This work provides further explanation as to why supply boom and bust patterns are observed in many localized real estate markets.

⁴ See Buttimer and Ott (2007) for a similar set-up for already built properties, but one without the determination of a market equilibrium, land valuation, time to build and optimal development timing.

level where land developers are willing to presell/prelease space preceding an ultimate development decision. The equilibrium cash flow level can then be mapped into an expected absorption of space over time.

We model the cash flow level and the tenant's/buyer's reservation cash flow as mean-reverting stochastic processes reverting to the endogenously determined long-run equilibrium. These processes reflect empirically observed economic cycles and slow real estate supply adjustments due to time to build. Specifying the correlation between the two uncertainties along with the elasticity of demand allows us to model development in varying local economic environments.

We solve the model numerically for a given set of input parameters to determine the real estate market's equilibrium cash flow, which is then used to compute, in closed-form, the market's equilibrium price per completed development unit. Based on this market equilibrium, we determine undeveloped land values, the preleasing/presales necessary to initiate development, and expected time to development for properties without the requisite presales. We are also able to determine the expected return to investing in undeveloped land. Perturbing the input parameters allows us to isolate how the change affects each of these values and determine those parameters that most affect the results.

Empirical evidence and investor surveys support the outputs of our model. For example, in the preponderance of the cases where parameters are varied, we find that expected returns on speculative land investment generally range from 12% to 25%. Risk and expected returns are affected primarily by completed space demand and absorption sensitivity, development timing risk and the degree of leverage inherent in the land

option. Absorption times to sell or lease any remaining units that are not presold/leased run 3 to 4 years on average. Absorption levels and times are relatively constant across different parameter sets as completed project prices adjust in equilibrium to smooth changes in absorption. Land values range from 12% to 25% of completed project value and presales needed to commence construction are generally 33% to 60% and increase with the level of risks inherent in the land.

In the following sections, we first develop the model and present the general economic environment in which it is placed. We demonstrate that we can solve this model using standard contingent-claims techniques. Solutions to the model for a variety of economic environments are then presented and the implications of the model are discussed. The analysis concludes with a discussion of our findings.

2. An Economic Model of Real Estate Markets

We develop a model applicable to land development and space markets in both commercial and residential settings. Each of these settings exhibits clear differences in the rate at which potential buyers arrive, per-unit construction costs, and the level of uncertainty regarding future sales prices.⁵ However, these differences are primarily parametric and the fundamental economic process of land development is the same for both settings. The commonality allows us to build a general development model.

2.1. Basic Features of Land Development

We begin by describing the general land development process. A landowner sells to a

⁵ We will use the term buyer from this point forward to refer to either buyers or renters (tenants) of space. Price will be used to refer to the price of space for sale or the rental price of space for rent, where rent flow maps directly into a space value function.

developer a piece of undeveloped land at time 0. The developer will, either at time 0 or at some point in the future, begin the process of developing N units on that land.⁶ The developer chooses when to begin construction. Once started, construction takes T^C months to complete.⁷ We assume that the developer will develop all N units simultaneously to maintain economies of scale.

The developer will seek to pre-sell units prior to and during the construction period with delivery of all pre-sold units upon completion.⁸ Buyers in the pre-construction phase place deposits on their units, but have the option to abandon their purchase (by giving up their deposit) should they chose to do so at any time prior to the beginning of construction. Once construction begins, however, they are committed to purchasing the unit. Buyers that purchase units after construction has begun do not get the option to terminate their purchase. Note that these purchasers will pay the then-prevailing market price for the unit at closing. A developer unable to pre-sell their entire inventory will close upon the remaining units at the end of construction at the spot market price, subject to stochastic buyer arrivals.

Before developing the model to determine land value, we must first develop several of the model's sub-components. These include features such as the property service flow, which maps into the space price function, and the developer cash flows. We define these components in the following sub-sections.

⁶ We use the somewhat vague term “units” intentionally. The model is flexible enough to handle cases where the “units” are developed lots which are sold to homebuilders or commercial developers, and also cases where the units are residential and commercial buildings that are rented and/or sold.

⁷ We do have to place a constraint on the developer's option to begin development: they must do so prior to reaching some development horizon, T^H . We set this development horizon far enough into the future where discounting effects will render it moot. Normally we set this to 15 years in our numerical solutions.

⁸ We use the term “pre-sold” for convenience, if the developer is creating rental space they would be pre-leasing units.

2.1.1. The Service Flow Processes for Built Space

The developer's return on investment is based primarily upon the price received for the units created. Ultimately the units' values are derived from the economic benefits they produce, i.e. the present value of the service flows (cash flows) they provide.⁹ In our model, the real estate market for *built space* is perfectly competitive where all market participants are price takers with respect to the equilibrium spot service flow rates.

First, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, where the $\{\mathcal{F}_t\}_{t \geq 0}$ (the information structure) is the \mathbb{P} -completion of a filtration generated by the vector stochastic process that we construct to model the evolution of the state of the system through time. Let M_t denote the price of the traded built property market index and assume that it follows,

$$dM_t = \delta_M M_t dt + \sigma_M M_t dz_M(t) \quad (1)$$

Where δ_M is the expected growth in cash flows (the capital gain component) of the index, $\sigma_M > 0$, and z_M is a standard Brownian motion under \mathbb{P} . Thus the market price of risk for the property index is $\phi = \frac{\alpha_M - r}{\sigma_M}$ where α_M is the total expected return.

The service flow provided by a specific building, however, is due to a combination of a local factor and the general property index. The local factor for an individual building is measured in units of M_t , which when multiplied by M_t gives the

⁹ The type of service flow will depend upon the context of the specific development we model. For example, in a leased building the service flow will be the net rental flow. In the case of an owner-occupied commercial building, the service flow is the stream of economic benefits which accrue to the owner from physically occupying the building. The service flow for owner-occupied residential housing is the implicit rent on the property.

service flow in dollars per unit time. We assume that this local process, denoted by P_t^L , satisfies the following mean-reverting process

$$dP_t^L = \kappa_{p^L} (\bar{P}_L e^{\alpha p t} M_t^{-1} - P_t^L) dt + \sigma_{p^L} P_t^L dz_{p^L}(t) \quad (2)$$

where $\kappa_{p^L}, \bar{P}_L, \sigma_{p^L} > 0$, z_{p^L} is a standard Brownian motion under \mathbb{P} with $dz_{p^L} dz_M = 0$.

Then the service flow process (measured in dollars) is $P_t = M_t P_t^L$ and

$$\begin{aligned} dP_t &= d(M_t P_t^L) = dM_t P_t^L + M_t dP_t^L \\ &= \delta_M M_t P_t^L dt + \kappa_{p^L} (\bar{P}_L e^{\alpha p t} M_t^{-1} M_t - M_t P_t^L) dt + \sigma_{p^L} M_t P_t^L dz_{p^L}(t) + \sigma_M M_t P_t^L dz_M(t) \\ &= \kappa_{p^L}' (\bar{P}_L' e^{\alpha p t} - P_t) dt + \sigma_M P_t dz_M(t) + \sigma_{p^L} P_t dz_{p^L}(t) \\ &= \kappa_{p^L}' (\bar{P}_L' e^{\alpha p t} - P_t) dt + \sigma_P P_t dz_P(t) \end{aligned} \quad (3)$$

where, $\kappa_{p^L}' = \kappa_{p^L} - \delta_M$, $\bar{P}_L' = \frac{\kappa_{p^L} \bar{P}_L}{\kappa_{p^L}'}$, $\sigma_P = \sqrt{\sigma_M^2 + \sigma_{p^L}^2}$, and $z_P(t) = \frac{\sigma_M z_M(t) + \sigma_{p^L} z_{p^L}(t)}{\sqrt{\sigma_M^2 + \sigma_{p^L}^2}}$

is a standard Brownian motion. We assume that the local spot service-flow factor follows a mean-reverting stochastic process to reflect a real estate market where time to build for new development creates an inelastic supply of built space in the short-term and service flows adjust (revert) to a long-run equilibrium over time.

Since there are two sources of risk in the service flow process, we must specify risk adjustments for both. Under the risk-neutral measure, the discounted price process for the market index must be a martingale since it is a traded asset. However, the local service flow process is not the price of a traded asset. Let \mathbb{P}^* be a probability measure equivalent to \mathbb{P} under which

$$dz_M^*(t) = \phi dt + dz_M(t), \quad (4)$$

and

$$dz_{p^L}^*(t) = \frac{\kappa_{p^L}' \bar{P}'_L (e^{\alpha_P t} - e^{(\alpha_P - \alpha_M + r)t})}{\sigma_{p^L} P_t^L} dt + dz_{p^L}(t) \quad (5)$$

Under \mathbb{P}^* ,

$$dz_p^*(t) = \frac{\sigma_M (dz_M^*(t) - \phi dt) + \sigma_{p^L} dz_{p^L}^*(t) - \kappa_{p^L}' \frac{\bar{P}'_L}{P_t^L} (e^{\alpha_P t} - e^{(\alpha_P - \alpha_M + r)t}) dt}{\sigma_P} \quad (6)$$

and

$$dP_t = \kappa_{p^L}' \left(\bar{P}'_L e^{(\alpha_P - \alpha_M + r)t} - \left(1 + \frac{\phi \sigma_M}{\kappa_{p^L}'} \right) P_t \right) dt + \sigma_P P_t dz_p^*(t) \quad (7)$$

If we set $\eta_P = \kappa_{p^L}' + \phi \sigma_M$, and $\bar{P} = \frac{\kappa_{p^L}' \bar{P}'_L}{\eta_P}$, then

$$dP_t = \eta_P (\bar{P} e^{(\alpha_P - \alpha_M + r)t} - P_t) dt + \sigma_P P_t dz_p^*(t) \quad (8)$$

We note that $dz_p^* dz_M^* = \frac{\sigma_M}{\sigma_P} dt$.

2.1.2 Completed Unit Values

A completed building unit, whether it is a lot, a house, an apartment, an office, or a retail facility, will be worth the present value of its expected future service flows over its economic life, T^B . The unit's value, denoted as B , at time t is given by:

$$B(t, T^B, P_t) = E_t^* \left[\int_t^{T^B} e^{-r\tau} P_\tau d\tau \right] \quad (9)$$

where E_t^* denotes conditional expectation with respect to \mathbb{P}^* . We can find the value of (9) by noting that Fubini's Theorem applies here allowing us to interchange the order of the integral and expectation. Thus,

$$\begin{aligned}
B(t, T^B, P_t) &= E_t^* \left[\int_t^{T^B} e^{-r\tau} P_\tau d\tau \right] = \int_t^{T^B} e^{-r\tau} E_t [P_\tau] d\tau \\
&= \frac{(\eta_p \bar{P} - P_t (\alpha_p - \alpha_M + r + \eta_p)) e^{-(r+\eta_p)(T^B-t)}}{(\alpha_p - \alpha_M + r + \eta_p)(r + \eta_p)} + \frac{\eta_p \bar{P} e^{(\alpha_p - \alpha_M)(T^B-t)}}{(\alpha_p - \alpha_M + r + \eta_p)(\alpha_p - \alpha_M)} + \frac{(\alpha_p - \alpha_M) P_t - \eta_p \bar{P}}{(r + \eta_p)(\alpha_p - \alpha_M)}.
\end{aligned} \tag{10}$$

Therefore, given a current spot service flow rate P_t , an economic life of the unit T^B , and some parameters from the market implicit rent process, we can fully determine the value of the unit in closed-form.

2.1.3 The Reservation Service Flow Process

Although market participants are price takers with respect to the spot service flows, purchasers are heterogeneous with respect to the reservation service flow at which they are willing to buy a unit from the developer. The distribution of buyers has an average reservation service flow $R(t)$, and each individual potential buyer/lessor has an idiosyncratic reservation service flow $\hat{R}(t)$. When the market service flow rate $P(t)$ is above an individual buyer's reservation service flow rate, $\hat{R}(t)$, it is not economically viable for the buyer to purchase the unit. The average reservation service flow also follows a mean-reverting stochastic process to reflect a general economy that is cyclical in nature but adjusts to a long-run equilibrium over time. As with the market service flow, we map general economic conditions into a buyer service flow process measured in units of M_t , denoted by R_t^L , which then determines the buyer reservation price for a particular unit. That is, as economic conditions strengthen, R_t^L buyer service flows and reservation prices for the property are increased, and vice versa for weaker economic conditions. We assume that R_t^L evolves according to

$$dR_t^L = \kappa_{R^L} (\bar{R}_L e^{\alpha_R t} M_t^{-1} - R_t^L) dt + \sigma_{R^L} R_t^L dz_{R^L}(t) \quad (11)$$

where $\kappa_{R^L}, \bar{R}_L, \sigma_{R^L} > 0$, z_{R^L} is a standard Brownian motion under \mathbb{P} with $dz_{R^L} dz_M = 0$, and $dz_{p^L} dz_{R^L} = \rho_L dt$. As above, the average service flow process (measured in dollars) is $R_t = M_t R_t^L$ and

$$dR_t = \kappa_{R^L}' (\bar{R}_L' e^{\alpha_R t} - R_t) dt + \sigma_R R_t dz_R(t) \quad (12)$$

where, $\kappa_{R^L}' = \kappa_{R^L} - \alpha_M$, $\bar{R}_L' = \frac{\kappa_{R^L} \bar{R}_L}{\kappa_{R^L}'}$, $\sigma_R = \sqrt{\sigma_M^2 + \sigma_{R^L}^2}$, and $z_R(t) = \frac{\sigma_M z_M(t) + \sigma_{R^L} z_{R^L}(t)}{\sqrt{\sigma_M^2 + \sigma_{R^L}^2}}$

is a standard Brownian motion.

Under the risk-neutral measure, \mathbb{P}^* , we have

$$dz_{R^L}^*(t) = \frac{\sigma_M (dz_M^*(t) - \phi dt) + \sigma_{R^L} dz_{R^L}^*(t) - \kappa_{R^L}' \frac{\bar{R}_L'}{R_t^L} (e^{\alpha_R t} - e^{(\alpha_R - \alpha_M + r)t}) dt}{\sigma_R} \quad (13)$$

and

$$dR_t = \kappa_{R^L}' \left(\bar{R}_L' e^{(\alpha_R - \alpha_M + r)t} - \left(1 + \frac{\phi \sigma_M}{\kappa_{R^L}'} \right) R_t \right) dt + \sigma_R R_t dz_R^*(t) \quad (14)$$

If we set $\eta_R = \kappa_{R^L}' + \phi \sigma_M$, and $\bar{R} = \frac{\kappa_{R^L}' \bar{R}_L'}{\eta_R}$, then

$$dR_t = \eta_R (\bar{R} e^{(\alpha_R - \alpha_M + r)t} - R_t) dt + \sigma_R R_t dz_R^*(t) \quad (15)$$

We note that $dz_R^* dz_M^* = \frac{\sigma_M}{\sigma_R} dt$, and that the average market service flow will be positively

correlated with reservation service flows where $dz_p dz_R = \rho dt = \frac{\sigma_M^2 + \sigma_{p^L} \sigma_{R^L} \rho_L}{\sigma_p \sigma_R} dt$. This

reflects increased (decreased) demand for units with higher (lower) reservation service

flows. In the long-term, reservation service flow shocks will adjust to equilibrium over time reflecting general economic cycles.

2.1.4 Buyer Arrival and Departure Processes

One of the key elements of this model is that it incorporates the lumpiness of buyer availability in the market for finished space. The model assumes that willing and able buyers arrive in the market stochastically, where the probability of a buyer arriving is a function of both the current market price and the reservation price of the average buyer in the market. Similarly, in the pre-construction period, buyers will also depart the market, i.e. give up their deposits and void their sales contracts, stochastically. Ultimately, our model requires a function which can calculate the conditional probability of there being m units available for sale at the end of the period, conditional upon having n units available at the beginning of the period. This function must incorporate the probability of potential buyers arriving and departing, the number of units available for sale, and the construction status of the building. We build this complex function from simpler functions below.

We begin with the notion that individual buyers who are willing and able to purchase units are those with individual reservation service flows, \tilde{R}_t , greater than or equal to the current market service flow P_t . They are assumed to arrive randomly according to a doubly stochastic Poisson process $q_A(t)$. We model the intensity, $\lambda_A(t)$, of this process at time t as a function of the average reservation flow R_r , and the market spot flow P_t :

$$\lambda_A(t) = \zeta_A \left(\frac{P_t}{R_r} \right)^{-\varepsilon_A} \quad (16)$$

The scaling function ζ_A allows us to simulate the different levels of responsiveness of the arrival process to market conditions. The constant elasticity demand parameter, ε_A , represents the responsiveness of arrivals to changes in P_t and R_r .

The probability of $k \geq 0$ willing and able potential buyers arriving in a time interval $t - s$, for $0 \leq s \leq t$, is

$$\mathbb{P}^* \{q_A(t) - q_A(s) = k \mid \mathcal{F}_s\} = e^{-\int_s^t \lambda_A(u) du} \frac{\left(\int_s^t \lambda_A(u) du \right)^k}{k!}. \quad (17)$$

During the pre-construction period we assume that buyers place deposits on to-be-constructed units, and that they have the option to void their contracts by giving up their deposits. We assume that the only circumstances under which this would happen is when the market service flow level rises above the purchaser's individual reservation service flow level, i.e. when $P_t > \tilde{R}_t$. We assume that for a given set of market conditions, P_t and R_r , this will occur according to a doubly stochastic Poisson process $q_D(t)$. We model the

mean rate of departures for the process at time t also as a function of the average spot price R_t and the market spot price P_t :

$$\lambda_D(t) = \zeta_D \left(\frac{P_t}{R_t} \right)^{\varepsilon_D}. \quad (18)$$

The variable ζ_D is a scaling variable that allows us to vary the responsiveness of the departure process to the market conditions.¹⁰ The probability of $k \geq 0$ buyers departing in a time interval $t - s$, for $0 \leq s \leq t$, is

$$\mathbb{P}^* \{q_D(t) - q_D(s) = k \mid \mathcal{F}_s\} = e^{-\int_s^t \lambda_D(u) du} \frac{\left(\int_s^t \lambda_D(u) du \right)^k}{k!}. \quad (19)$$

We assume that once construction starts the buyer is obligated to purchase the unit at the end of construction. In essence this means that the developer only faces purchaser departures in the pre-development phase of the project. We note that arrivals and departures are correlated through equations (16) and (18), however we do not posit causality between arrivals and departures.

Individually $q_A(t)$ and $q_D(t)$ may create degenerate results because they do not incorporate project-specific information which is available. For example, they will indicate positive probabilities of arrivals when there are no units remaining to be sold, or positive probability of buyer departures when no units have yet been pre-sold. As a result, it is more useful to incorporate these functions into a more general function which does incorporate project-specific information such as the number of units available for sale, and whether construction has begun on the project. This general function will be denoted

¹⁰ As with the arrival process, we assume that this jump process also represents non-systematic risk. We also suppress the $R(t)$ and $P(t)$ parameters and simply denote $\lambda_D(R(t), P(t))$ as $\lambda_D(t)$.

as $q(m | n)$, and will tell us the probability of having m units remaining to be sold at the end of the period conditional upon having had n units available at the beginning of the period.

From a practical standpoint, we actually create two versions of $q(m | n)$, one for use when the project is in the pre-construction period, and one for use once construction has begun or has been completed. For ease of exposition we first consider the simpler of the two versions, that which is used in the post-construction state of the project.

Since all sales after the beginning of construction are final, the model can ignore buyer departures in this version of $q(m | n)$. This means that by definition, the $q(m | n) = 0$ for $m < n$. Further, we note that the developer can never sell more than the N units which constitute the project. Since $q_A(t)$ is not limited to N , we truncate its distribution and include in the case where the aggregate number of arrivals exactly equals the number of available all of the remaining probability. Given this we can write a compound function for $q(m | n)$ for use in the post-construction state as:

$$q(m | n) = \begin{cases} 0 & \text{for } m > N \\ \left(1 - \sum_{i=0}^m q_A(i)\right) & \text{for } m = N \\ q_A(m - n) & \text{for } n \leq m < N \\ 0 & \text{for } m < n. \end{cases} \quad (20)$$

The pre-construction version of $q(m | n)$ is complicated by the fact that buyer departures are allowed. The model must thus calculate the joint probability of a given number of purchasers arriving and departing, and so we must construct the joint

probability density function of the two Poisson processes. We can determine the joint density function from the individual marginal densities. That is, we can determine the joint probability that A potential purchasers will arrive and D purchasers will depart, resulting in a net change of $A - D$ purchasers, by first determining the probability of A arrivals using equation (17) and D departures using equation (19), and then multiplying those probabilities together to get the joint probability of their occurring together. Adding together the joint probabilities of all potential arrival and departure levels which result in a net change of $A - D$ purchasers determines the cumulative probability.

We can once again impose some additional structure on the function by appealing to the practicalities of the development process. Specifically, we note that the aggregate number of departures, D , cannot exceed the number of units pre-sold at the beginning of the period, n . Similarly, regardless of how many new potential purchasers arrive the developer cannot sell more than the N units that will constitute the project. Since the Poisson probability equations (17) and (19) can generate probabilities for any number of arrivals or departures, we truncate those distributions to conform to these realities. Thus we include in the case where the aggregate number of departures exactly equals the initial number of pre-sales the aggregate probability from all cases where the aggregate number of departures would be greater than or equal to n . Similarly we include in the case where the aggregate number of arrivals would exactly equal the total number of remaining unsold units in the development the aggregated probability from all cases where the number of arrivals would exceed that capacity. We denote the aggregated probability of having m units pre-sold at the end of a pre-construction period, conditional on having n

sales at the beginning of the period as $q^*(m|n)$. The formula for determining $q^*(m|n)$ is:

$$q^*(m|n) = \begin{cases} 0 & \text{for } m > N \\ \sum_{i=0}^n \left[q_D(i) \left(1 - \sum_{j=0}^{(i-1)+(m-n)} q_A(j) \right) \right] + \left(1 - \sum_{i=0}^n q_D(i) \right) \left(1 - \sum_{i=0}^m q_A(i) \right) & \text{for } m = N \\ \sum_{i=\min(0, m-n)} q_D(i) q_A(i+(m-n)) + \left(1 - \sum_{i=0}^n q_D(i) \right) q_A(m) & \text{for } m < N \\ 0 & \text{for } m < 0 \end{cases} \quad (21)$$

where q_A and q_D are defined in equations (17) and (19), respectively.

2.2. Development Project Cash Flows

Having specified dynamics for the evolution of the market service flow, P_t , average reservation service flow R_t , and buyer arrival and departure, we can begin to discuss the cash flows that the developer will receive. These cash flows will vary depending upon where the project is in the development cycle. We can consider the cash-flows in four distinct regimes: pre-construction, during construction, at construction completion, and post-construction.

2.2.1 Pre-Construction Cash Flows

We assume that prior to construction the developer faces some minimal level of fixed costs each period. These costs represent basic ownership costs such as property taxes, hazard insurance, and basic property-rights enforcement. We assume that these costs are a function of the size of the property and as a result represent them as being a function of the number, N , of units that can be built on the property. That is, we actually specify a

per-unit cost, $C^{PC}(t)$, growing deterministically at the rate α_v , and then assume that the total cost to the developer is the product of that cost and the number of potential units they could build. Thus the costs to the developer of holding the undeveloped land each period as:

$$Cost^{Predevelopment} = N * C^{PC}(t) \quad (22)$$

2.2.2 Construction Cash Flows

During construction the developer will face a different set of costs. We denote the construction period as T^C months. We assume that in each period the developer must pay $1/T^C$ of the total construction costs $K(t)$, where K grows deterministically at the rate α_K . The developer must also continue to pay monthly holding costs for the property. The developer's cost in a given construction period is given by:

$$Cost^{In-development} = N * C^{PC}(t) + \frac{K(t)}{T^C}. \quad (23)$$

2.2.3 Cash Flows upon Construction Completion

Since development takes T^C time steps to complete, at $T^C + 1$ time steps after the commencement of construction the developer and purchasers close on all pre-sold units. With that closing, the developer no longer bears any costs for the sold units. The developer does, however, continue to pay holding costs on any unsold units. Again we recognize that these per-unit holding costs may differ from those borne in the pre-

construction and construction periods, and we denote them as $C^{CC}(t)$.¹¹ Thus, the developer's cost in the period when construction is completed is given by:

$$Cost^{Construction\ Completion} = (N - n^{presold}) * C^{CC}(t), \quad (24)$$

The developer's net income for the period in which construction is complete and in which pre-sold units are closed is given by

$$\theta_n^{Construction\ Completion} = n^{presold} * B(t) - (N - n^{presold}) * C^{CC}(t), \quad (25)$$

where $B(t)$ is determined by equation (10).

2.2.4 Post-Construction Cash Flows

In the periods after construction, the developer continues to pay holding costs on any units that are unsold. If a sale is made the developer receives the value of the unit, $B(t)$, as determined by equation (10). Thus, given n^{spot} sales in a given period and n^{closed} units closed (including pre-sales) in all previous periods, the developer's net-revenues are given by:

$$\theta_n^{Post-development} = [n^{spot} * B(t)] - (N - n^{closed}) * C^{CC}(t). \quad (26)$$

We now have in place all of the pieces needed to model the developer's options, and ultimately to determine the value of the undeveloped land to the developer. In the next section we develop the formal mathematics of the model.

2.3. The Stochastic Control Model

The problem facing the developer is choosing the optimal construction start time; once started we assume that construction must continue for a deterministic length of time. Due

¹¹ One can easily justify this by noting that property taxes rates and insurance rates may differ for vacant land, property that is pre-completion, and completed properties.

to this feature, the problem is not one of “classical” stochastic control since the optimal constrained control would not be Markovian (once turned on, it cannot be turned off for a deterministic length of time, even if it is optimal to do so). The problem can, however, be formulated as an optimal switching problem as in Brekke and Øksendal (1994), which is in the class of optimal stopping problems.

To do this we augment the state process to include a state that is 0 if construction has not yet started and 1 if construction has started (including any time during construction and after completion). Since the construction time and costs (given the start time) are deterministic, the present value of construction costs at the start time is the switching cost, i.e. the cost of using the impulse control. Once formulated as a stochastic impulse control problem, it can be solved (with some minor modifications) using the methods of stochastic impulse control for jump diffusions as detailed in Øksendal and Sulem (2005). We present a formal derivation of the stochastic control model in Appendix A.

Unfortunately there is not a closed-form solution due to the highly path-dependent nature of the problem. As a result, we use a numerical method to determine the solution, which we discuss in the following section. In Appendix B we show that the numerical method provides a good discrete approximation to the value function.

2.4 The Numerical Method

Our numerical method must model the evolution of P_t and R_t through time. We chose to do this through a bivariate-binomial lattice using the method of Nelson and Ramaswamy (1990). We provide full details of the lattice construction in Appendix B. We use this lattice to solve the valuation equation subject to a set of terminal and interior boundary

conditions. We discuss those conditions, as well as various interior boundary conditions in the sub-sections that follow.

Given the stochastic control model, we can now write the valuation equations used within the lattice that satisfies those conditions. These equations also allow us to find the free-boundaries within the lattice of the various options held by the developer. In some sense the free-boundary that we are most interested in is the one associated with the developer's decision to begin construction. The existence of this boundary illustrates a path-dependency problem for the model.

Since the model works backwards in time, at an arbitrary point in time t we cannot know for certain whether the developer has already exercised their option to begin construction.¹² This means that we cannot know whether the project is in its pre-construction period, during one of the T^C months of its construction, at the completion of its construction, or in post-construction. Thus, there are a total of $T^C + 4$ potential states that the project could be in at time t .

Fortunately, there is a numerical technique we can use to solve this path-dependency problem. The key is to recognize that the uncertainty with respect to the building state and entitlement state at time t resolves as the model moves backwards in time. For example, at time $t - 1$ we know whether the builder would elect to begin construction at that time period. Once we know this we also know whether it is possible for the project to be in its first month of construction at time t , since if construction does not begin at $t - 1$, it cannot be one month along at time t . In fact, all uncertainty regarding the building state resolves in a similar manner as the model moves backwards in time.

¹² We will demonstrate shortly that for t near the development horizon T^H , we sometimes do know whether the developer has exercised that option.

We can exploit this resolution of uncertainty to solve the path-dependency problem. We do this by artificially expanding the state space to include new temporary state variables: construction status. That is, at an arbitrary point in time t we determine what the value of the project would be for each of the possible $T^C + 4$ construction states in which the project could be at that time. As the model moves further back in time the uncertainty regarding the time t construction state resolves and we can begin to discard some of these temporary state variables. Ultimately we will reach a point, potentially the initial point of the lattice, in which all building-state uncertainty within the lattice is resolved, and we retain only the correct time t state variable.¹³

We note that at the beginning and end of the model the path-dependency problem is somewhat lessened. For example, at time 0 we know with certainty that the project is in a pre-construction state. At time step 1, there are only two potential states of the project: it is still in pre-construction, or it is in the first month of construction. Only at time step $T^C + 2$ and beyond are all $T^C + 4$ construction states possible. Similarly, at time T^H , the end of the economic horizon of the developer (and the end of the model), and only three construction states are possible: the project could have been completed at some point in the past, it could have been completed during this time period, and it could still be in a pre-construction status. If the project is tagged as being in pre-construction this means that the developer has previously abandoned the project. The model does not allow the developer to begin construction on the project if completion would occur after the

¹³ We note that the problem is actually more complex because there is also uncertainty regarding how many units remain to be sold (or pre-sold.) That is, for each of the T^C+4 building states, there are $N+1$ potential sales states of the project. These states include the situation where there are 0 units remaining, 1 unit remaining, 2 units remaining, etc. Thus, at most nodes in the lattice there are actually $(N+1)(T^C+4)$ conditions (and hence temporary state variables) to be considered. Although we suppress the sales-states in this discussion for expositional clarity, the model does fully consider these conditions.

economic horizon of the developer, T^H . Thus, at time $T^H - T^C$ the developer is forced to make a decision, they must either begin construction or abandon the project.

In the sections that follow we develop the valuation equations that apply at each of the potential states of the project. We subsequently then combine those valuation rules to develop the decision rules that the developer uses to determine the project's value.

2.4.2 Post-Construction Interior Boundary Conditions

We begin our discussion of interior boundary conditions by introducing some additional notation. The value of the project at any node in the lattice (i, j) at time step t will depend upon the number of units which have already been sold, v , and we denote this value as $V_{t,i,j}(v)$.¹⁴ We also need to reference the time t expected value of the unit at time $t + 1$ under the risk-neutral probability measure. This value is also conditional upon the number of units sold after time $t + 1$, and we denote it as $\tilde{V}_{t,i,j}(v)$. Note that the t in the subscript indicates that this is the discounted expected value at time t of the project at time $t + 1$. The boundary condition given in equation (23) requires that $\tilde{V}_{T^H,i,j}(v) = 0$ for all values of n since T^H is the terminal point of the lattice.

In any post-construction period, the actions available to the developer will depend upon the number of unsold units still available. If all units have been sold, the remaining value of the project $V_{t,i,j}$ will be zero, since there are neither costs to be born nor additional revenues to be earned. Should there be units available, however, the developer must decide whether they are willing to sell those units this period should potential purchasers arrive. If the developer sells those units today they give up the discounted

¹⁴ When referencing specific points in our lattice, we must use a notation with three coordinates. This is because the lattice contains three primary state variables: P, R and time.

expected present value (under the risk-neutral probability measure) of the project at the next time step. The developer also has the option to abandon any unsold units. Again, the developer must weigh the benefit of doing this against the cost of forgoing any future sales of those units. If the developer does abandon, however, they gain the benefit of not paying holding costs. This is given by:

$$V_{t,i,j,\chi=T^C+1}(n) = \sum_{m=0}^n \left\{ q^*(m|n) * \max \left(\begin{array}{l} \theta_m^{post-development} + \tilde{V}_{t,i,j,\chi=T^C+1}(m), \\ \theta_n^{post-development} + \tilde{V}_{t,i,j,\chi=T^C+1}(n), 0 \end{array} \right) \right\} \quad (27)$$

Note that this equation determines the value of the project across all potential number of unit sales and allows the developer to determine exactly how many units they are willing to sell. These different values are weighted by the probability of the different numbers of purchasers ($m-n$) arriving in that period.

2.4.3 Construction Completion Boundary Conditions

The builder faces a very similar decision if construction is completed in the current time step. They will receive, with certainty, revenues for units that have been pre-sold. They will also receive revenues for any units that they sell in the current period. They have to weigh selling those units in the current period against the possible benefits of delaying sales, i.e. the expected value of those future sales under the risk-neutral measure.

Similarly, the developer could elect to abandon the unsold units at time t . Thus, there are three options for the developer to consider: selling units in this period, refusing to sell units in this period but continuing to operate the project, and abandoning the project entirely. The developer will elect the option that maximizes their wealth. We can incorporate all three options into a single equation as:

$$V_{t,i,j,\chi=T^c}(n) = \sum_{m=0}^n \left\{ q^*(m|n) * \max \left(\begin{array}{l} \theta_m^{Construction\ Complete} + \tilde{V}_{t,i,j,\chi=T^c+1}(m), \\ \theta_n^{Construction\ Complete} + \tilde{V}_{t,i,j,\chi=T^c+1}(n), 0 \end{array} \right) \right\} \quad (28)$$

The three terms within the max function correspond to the sell, no-sell, and abandon options. Also, recall from equation (20) that the net income function at completion ($\theta^{Construction\ Complete}$) requires two inputs – the number of units pre-sold and the number sold in the current period. Since the total number of units in the development are N, and if there are n remaining at the beginning of the time step, this implies that (N-n) are pre-sold, and so the m that are sold during the period are spot market sales.

2.4.4 During Construction Boundary Conditions

During the construction period the developer has the option of selling units if potential purchasers arrive.¹⁵ Once again, the developer must weigh the value of pre-selling immediately versus the value that they give up of being able to sell in the future. The developer also continues to have the option to abandon the project. If they abandon, they not only avoid future holding costs, but they also avoid the remaining construction costs that they have yet to bear. The developer will once again take the actions, conditional on the number of arrivals of potential purchasers that maximizes their position. We represent this through the following equation.

$$V_{t,i,j,\chi=k}(n) = \sum_{m=0}^n \left\{ q^*(m|n) * \max \left(\begin{array}{l} \theta_m^{In-Development} + \tilde{V}_{t,i,j,\chi=k+1}(m), \\ \theta_n^{In-Development} + \tilde{V}_{t,i,j,\chi=k+1}(n), 0 \end{array} \right) \right\} \quad (29)$$

Again, the three terms in the max function correspond to the three options held by the developer. It is worth noting that the $\tilde{V}_{t,i,j,\chi=k+1}(n)$ and $\tilde{V}_{t,i,j,\chi=k+1}(m)$ values referred to in equation (34) are based upon the assumption that at time t+1 the construction is one-

¹⁵ Technically these are pre-sales since they will not close until construction is completed.

period further along, so $0 < k < T^C$.¹⁶ If we are in the last time period of construction, these expected values will be based upon the completed construction cash flows, i.e. equation (33), at the next time step.

2.4.5 Pre-Construction Boundary Conditions

During the pre-construction period, the developer holds multiple options. These options interact to create a number of unique boundaries. Specifically we note that the developer has five specific actions that they can take:

1. They can delay construction but pre-sell to arriving potential buyers;
2. They can delay construction and refuse to pre-sell to arriving potential buyers;
3. They can begin construction and pre-sell to arriving potential buyers;
4. They can begin construction and refuse to pre-sell to arriving potential buyers;
5. They can abandon the project.

If the developer delays construction, then the expected present values used in the valuation equations are conditional upon the assumption that the next period will also be a pre-development period. If the developer begins construction, then the expected present values are conditional upon the assumption that the next period will be month 1 of the construction process, and that the project will be complete in T^C months. The developer will choose the option that maximizes their position. We can express this in the following equation:

¹⁶ An implication of this is that we do not allow the developer to halt development and the recommence it. Once development has started the developer can only continue to develop or abandon the project.

$$V_{t,i,j}(n) = \sum_{m=0}^n q(m|n) * \max \begin{cases} \theta_m^{Pre-Development} + (\tilde{V}_{t,i,j}(m) | \text{delay}), \\ \theta_n^{Pre-Development} + (\tilde{V}_{t,i,j}(n) | \text{delay}), \\ \theta_m^{Pre-Development} + (\tilde{V}_{t,i,j}(m) | \text{start}), \\ \theta_0^{Pre-Development} + (\tilde{V}_{t,i,j}(n) | \text{start}), \\ 0 \end{cases} \quad (30)$$

Note that equation (30) allows us to determine the value of the land. To see this, consider that every other equation ultimately feeds into this equation. At time step 0, with 0 units pre-sold (i.e. with N units yet to be constructed and sold) this is the most that the rational developer would pay for the land in order to begin the development process.

2.5 Finding Equilibrium Values

We can also solve the system numerically to determine the market's equilibrium service flow value, which in turn is used to compute the market price per completed unit. We derive the equilibrium service flow as the level where land developers are willing to presell/prelease space preceding their ultimate development decision. Although it is assumed that there is finite price elasticity for the real estate market at the aggregate level, it is also assumed that individual land parcels are perfect substitutes, so that prices above the market's equilibrium of $P_0^L = \bar{P}_L$ are not obtainable by individual properties due to competing land supply.

The equilibrium service flow value is determined by finding the $P_0^L = \bar{P}_L$ value, for properties with no endowed pre-sales as of time 0, where the value of the land is maximized¹⁷

$$\left. \frac{\partial V_{t,i,j,\chi=0}}{\partial P_0^L} \right|_{n=0, P_0^L = \bar{P}_L} = 0 \quad (31)$$

Properties without presales maximize land value at the *lowest* $P_0^L = \bar{P}_L$ relative to land value with *any* number of presales. Properties with presales will not be allowed to raise prices beyond this point since land owners without presales can maximize their land value and presell units at a lower price; thus, these properties are the price setters in the market and this is where the market price will settle.

2.6 Expected Returns and Time to Development

As shown in Appendix B, it is possible to recover the risky transition probabilities at each node in the lattice. This allows us to examine both the price of the property, which is governed by the risk-neutral probabilities, as well as the actual expected returns and time until development to the property, which are governed by the risky probabilities.

To compute these, we first solve for the price of the land working backwards in time. As we do so, however, we store the cash flows and actions of the developer under the risk-neutral measure for every state at every node.¹⁸ After determining the value of

¹⁷ As $P_0^L = \bar{P}_L$ is varied there are two opposing effects: for increases (decreases) in $P_0^L = \bar{P}_L$, the market value of land increases (decreases) because of the increased (decreased) value of the competed unit; while, at the same time increases (decreases) in $P_0^L = \bar{P}_L$ reduce (increase) the number of arrivals due to finite supply and demand elasticity for the market as a whole. Given these competing effects, there is a $P_0^L = \bar{P}_L$ value for properties with no presales where land value is maximized.

¹⁸ This does require a tremendous amount of memory – even with 8 gigabytes of memory, at best we can only value a 16 unit building with at most a 36 month construction period when using 4 time steps per month.

the project using the risk-neutral probabilities, we work our way forward through the lattice using the risky-probability measure. At each node we examine the cash flows and actions taken by the developer. We then use the risky probabilities to determine the actual expected cash flow from the project. From this, we can calculate the expected return as the IRR of the project, given the price determined by the risk-neutral valuation. Similarly, by noting where in the lattice the developer exercises their development option, we can then use the risky probabilities to determine the expected time until development in the risky world.

By calculating price, expected return, and time to development, we are able to demonstrate the degree to which the model is able to replicate behaviors seen in the real world. In the next section we present the basic results of the model.

3. Results from the Model Simulations and Analysis

3.1 Base Case Parameters

Using a set of parameter value inputs and the specified numerical procedures described, we solve the model and compute a number of output values: equilibrium local service flow $P_0^L = \bar{P}_L$ (which then maps into the equilibrium completed building value using equation (10)), the value of the speculative land (land without presales), the expected time until speculative land will begin construction, the expected return (IRR) on speculative land until construction commences, the value of land ripe for development (i.e., the land with the presales necessary to justify immediate commencement of construction), land value as a percentage of the completed project, the number of presales needed to promote construction commencement (which then maps into the equilibrium level of absorption of units per year and the expected years to fully absorb the remaining

space).

Table 1a presents the parameters for our base-case scenario. First we specify the initial market price of the general property market index M_0 as the numeraire and set its total return, drift rate (capital gain component) and volatility to be 10%, 3.5% and 12.5% respectively.¹⁹ The parameters that relate to the local factors of the service flow and the reservation service flow are those that populate the stochastic processes for service P_t^L and R_t^L . $R_0^L = \bar{R}_L$ is initially set to be the numeraire, while the equilibrium $P_0^L = \bar{P}_L$ is found endogenously. Since we would expect the market and reservation service flow to behave similarly over the long term, we set both processes drift terms, reversion speeds, and volatilities of to be 3.5%, 12.5% and 15% respectively. We set the correlation coefficient ρ_L between the local the market and reservation service flow processes to be 0.25. The risk-free rate (r) is set to 5%.

Table 1a:	
Base Case Model Parameters with respect to the general economic conditions	
Unless otherwise noted, parameters are quoted in an annualized format.	
Parameter	Base Case Value
α_M is the expected total return on the market portfolio	$\alpha_M = 0.10$
δ_M is the capital gain component of the market portfolio's return	$\delta_M = 0.035$
σ_M is the constant volatility rate on the market portfolio return	$\sigma_M = 0.125$
α_P is the drift of the service flow	$\alpha_P = .035$
η_p is the speed of reversion of the service flow	$\eta_p = .15$
σ_P is the constant volatility rate of the service flow	$\sigma_P = .125$
α_R is the drift of reservation service flow	$\alpha_R = .035$
η_r is the speed of reversion to the reservation service flow	$\eta_r = .15$
σ_R is the constant volatility rate of the reservation service flow	$\sigma_R = .125$
ρ_L Correlation between the service flow price and the reservation service flow	$\rho_L = .25$
$R_0^L = \bar{R}_L$ is the initial reservation service flow value	$R_0^L = \bar{R}_L = 1$
r is the risk-free rate	$r = .05$

¹⁹ The level of these inputs is based on common real estate return indices such as the National Council of Real Estate Investment Fiduciaries the National Association of Real Estate investment Trusts, among others. Also see The Benefits of Real Estate (2006) authored by the Center for International Securities and Derivatives Markets.

In Table 1b, we specify the parameters for the particular development project. The completed development project will be a total of $N=16$ units.²⁰ The developer is endowed with pre-sold units at time 0 anywhere from $n=0$ up to $n=16$. For the buyer arrival function, we specify the base case scalar of arrivals ζ_A to be 7. The elasticity of arrivals for new unit buyers ε_A is set to be 6.²¹ For the departure function, we specify the base case scalar of departures ζ_D to be 1 and the elasticity of departures ε_D to be 3. The elasticity of departure is set to one-half of arrivals to reflect a level of hysteresis for exiting a contract to purchase a unit. With these parameters, when P_t and R_t are equal the average number of arrivals of buyers to the project per year is 7 and the number of departures is 1.²²

The initial annual costs on a per unit basis of holding the land until construction is complete is $C_v = \$0.0425$. This represents approximately 3% per year of preconstruction speculative land value (land with zero presales). The annual cost of holding an unsold unit after construction is complete is $C_v = \$0.50$. This again represents approximately 3% per year of the value of the unsold but *completed* unit. Both of these costs will set to increase at a rate of 3% per year. Also shown in Table 1b are the base-case total construction costs of \$12.50 per unit and the time to complete construction of $T^C = 18$

²⁰ We choose 16 due to computing capacity.

²¹ Although the elasticity may seem high, it has been found that elasticities for *new* residential construction are historically between 4 and 13. See Malpezzi and Maclennan (2001) and Green, Malpezzi and May (2005).

²² See equation (16) and equation (18) where the arrival and departure Poisson process are defined and explained. With a 16 unit project, if the P/R ratio is one, the base case parameters provide for an expected absorption for the project of between 2 and 3 years (16 units/7 net arrivals per year), which is realistic in today's market for many residential and commercial projects.

months. Construction costs are assumed to increase at 3% per year. Once a unit is constructed it is assumed that it has a useful life of $T^B = 50$ years.

Table 1b:	
Base case model parameters with respect to the development project	
Unless otherwise noted, parameters are quoted in an annualized format.	
Parameter	Base Case Value
N, number of lots/units in the development project	N = 16
n, number of presold units endowed at time 0	n = 0, 1...16
ζ_A is the arrival scalar	$\zeta_A = 7$
ζ_D is the departure scalar	$\zeta_D = 1$
ε_A is the arrival elasticity	$\varepsilon_A = 6$
ε_D is the departure elasticity	$\varepsilon_D = 3$
T^C time in months to complete construction once development begins	18
T^B is the useful life of a completed building	50
K is the total construction cost for unit	\$12.50
α_K is the growth rate for construction costs over time	$\alpha_K = .03$
$C_v(0)$ is the initial variable cost per unit <i>upon</i> construction completion	$C_v = \$0.50$
$C_v(0)$ is the initial variable cost per unit <i>before</i> construction completion	$C_v = \$0.0425$
α_v is the drift rate for variable costs over time	$\alpha_v = .03$

3.2 Numerical Results and Comparative Statics

Given the inputs shown in Tables (1a) and (1b), we run the model to determine the outputs. In Table 2 we show the results for the base-case parameters (shown in bold) along with comparative statics for varying the elasticity of arrival parameter ε_A . We find that speculative land value is maximized at a local service flow price of $P_0^L = \bar{P}_L = 1.180$ which maps into an equilibrium market price of *each* of the completed building units of \$17.45. The value of speculative land, on which 16 total building units could be constructed, is \$11.43 and it is expected to begin construction in 66 months. The expected return (IRR) on an investment in speculative land until the uncertain time of construction is computed to be 18.17%. Land that is ripe for development, i.e., with presales of 38% of the units, land value has increased to \$35.07, representing 12.56% of

the completed project value, and expected absorption of the remaining units will occur over the next 3.86 years (this includes any absorption during the 18 month construction period). All of these results are in the range of reasonableness based on a number of empirical observations and sources.²³

Table 2 also reports results where the arrival price elasticity, ε_A in the demand function Equation (16), is varied from the base case. The elasticity of arrival parameter is instrumental not only in determining the arrival volatilities of the demand function, but also in finding the equilibrium local service flow $P_0^L = \bar{P}_L$, which is endogenously determined using the speculative land supply optimization equation (31). Where the price elasticity is less than the base case, equilibrium service flows, completed unit prices and land values all increase. Decreased elasticity affects the equilibrium service flows through equation (31) in the sense that it reflects markets that are less competitive due to low land *supply* elasticity, e.g., in markets where land is constrained by public policy and/or geographic reasons. In these less competitive land markets, demand is less affected by economic changes, completed project and land values are higher, development on available land occurs sooner, land values are a larger percentage of completed project value, and expected returns from land investment are lower due to decreased market risk from less demand/absorption uncertainty. These results reverse for more competitive markets where land development is less constrained and supply elasticities are higher.

²³ See Davis and Heathcote (2007), Davis and Palumbo (2008), Korpacz Real Estate Investor Survey[®] (2nd quarter, 2008) and Zelman, McGill, Speer, and Ratner, 2006.

Table 2
Summary of Model Outputs as a Function of Arrival Elasticities

Column	A	B	C	D	E	F	G	H	I	J
Arrival Elasticity $\varepsilon_{AP} = \varepsilon_{AR}$	Equilibrium Service Flow $P_0^l = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
2	2.269	33.55	115.61	9	7.05	197.66	36.82%	25%	1.36	8.83
3	1.524	22.54	45.40	11	9.65	76.91	21.33%	19%	1.98	6.57
4	1.299	19.21	24.14	29	13.56	50.16	16.32%	25%	2.46	4.88
5	1.217	18.00	15.59	50	16.36	39.75	13.80%	31%	2.62	4.20
6	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
7	1.160	17.15	9.13	78	19.34	32.96	12.01%	44%	2.48	3.63
9	1.150	17.01	7.77	86	20.03	32.48	11.94%	44%	2.29	3.93
9	1.142	16.89	7.03	93	20.52	33.08	12.24%	50%	2.12	3.78
10	1.140	16.86	6.89	98	20.65	28.68	10.63%	56%	1.89	3.71
11	1.134	16.77	5.95	100	20.95	31.59	11.77%	63%	1.76	3.49
12	1.132	16.74	5.70	103	21.01	33.65	12.56%	69%	1.58	3.16

Base Case outputs are shown horizontally in bold. Note that departure elasticities are set at 50% of arrival elasticities.

Columns A and B: The equilibrium local service flow maps into the equilibrium completed building value using equation (15).

Column C: The value of speculative land represents the value of land without presales.

Columns D and E: The expected time to development is computed as the expected number of months before construction will begin on speculative land.

Expected return (IRR) is computed based on the expected cash flows from an investment in speculative land from time zero until construction commences.

Column F and G: The value of land ripe for development represents land with the presales necessary to justify commencement of construction. Land value that is ripe for development is shown as a percentage of the completed project value.

Columns H, I and J: The percentage of units to be presold is the amount needed to commence construction, which then maps into equilibrium level of absorption of units per year and the expected years to fully absorb the remaining space post construction.

Table (2) indicates that most results are extremely sensitive to changes in elasticity with exception that changes in the equilibrium completed unit prices tend to smooth absorption levels across different levels of elasticity. An exception occurs for markets with very low elasticities, where very high prices are available allowing for tolerance of slower absorption levels.

Risk and expected returns are affected in two major ways as elasticity is varied. First, with lower (higher) elasticity, demand is less (more) sensitive to changes in reservation and price service flow levels, mitigating (exacerbating) demand/absorption volatility and risk. Secondly, because land is an option to develop completed units by paying the construction costs (the exercise price), changes in the equilibrium completed unit values directly affect the level of leverage inherent in the land option, i.e., the extent the option (land) is in- or out-of-the -money. In less (more) competitive markets with lower (higher) demand and supply elasticities, the construction costs represent a smaller (larger) proportion of the completed project value and the inherent option leverage and risk is decreased (increased). These two sources of risk will be a common theme throughout the presentation of the results.

The sensitivity of many of the outputs to elasticity in Table (2) shows the importance of determining demand and supply elasticity parameters more accurately through additional econometric studies and/or the through the calibration of other more transparent parameters within the model so as to imply the market's elasticity.

We now examine the effects of increases in the duration and cost of construction. Panel A of Table 3 shows that increasing construction costs increase the necessary equilibrium market price of completed units and at the same time decrease land values

quite significantly. The *percentage* effect of construction cost on land values is leveraged upward since land values typically represent only 12-25% of the completed development project value. Additionally, increases in construction cost extend development timing risk and increase inherent option leverage causing expected speculative land returns to increase. This increase in risk leads to an increase in the level of presales needed to begin construction.

Panel B of Table 3 shows similar results when construction time is varied. As construction time increases, it indirectly increases the overall net cost of the project, as completed presold units cannot be sold and revenues collected until the project is completed. As the cost of this delay increases, the necessary market price of completed units increases. Unlike the construction cost results where the leverage of the option is affected by the direct increase in construction costs (exercise price), expected returns decrease with increased construction time since exercise price is constant and completed unit market prices increase. All of the results from Table (3) show that increased costs in construction are partially passed forward to the price of completed units and partially passed backward to decrease land values.

Table 3
Summary of Model Outputs as a Function of Constriction Costs and Construction Time

Panel A: Construction Costs										
Column	A	B	C	D	E	F	G	H	I	J
Construction Costs	Equilibrium Service Flow $P_0^t = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
180	1.113	16.46	24.42	21	13.28	42.72	16.22%	25%	3.68	3.26
200	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
220	1.286	19.02	3.82	123	22.47	38.78	12.74%	56%	1.55	4.52

Panel B: Construction Time										
Construction Time	Equilibrium Service Flow $P_0^t = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
12	1.163	17.20	13.43	64	17.94	37.25	13.54%	38%	2.83	1.163
18	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	1.180
24	1.201	17.76	9.31	72	18.44	32.42	11.41%	38%	2.33	1.201

Base Case outputs are shown horizontally in bold.

Columns A and B: The equilibrium local service flow maps into the equilibrium completed building value using equation (15).

Column C: The value of speculative land represents the value of land without presales.

Columns D and E: The expected time to development is computed as the expected number of months before construction will begin on speculative land. Expected return (IRR) is computed based on the expected cash flows from an investment in speculative land from time zero until construction commences.

Column F and G: The value of land ripe for development represents land with the presales necessary to justify commencement of construction. Land value that is ripe for development is shown as a percentage of the completed project value.

Columns H, I and J: The percentage of units to be presold is the amount needed to commence construction, which then maps into equilibrium level of absorption of units per year and the expected years to fully absorb the remaining space post construction.

Table (4) shows comparative statics for changes in the local reservation service flow. Panel A of Table (4) assumes that the initial reservation service flow *and* the long-run reservation service flow are equal, i.e., $R_0^L = \bar{R}_L$. For cases where $R_0^L = \bar{R}_L$ is less (more) the base case, $R_0^L = \bar{R}_L = 1$, this reflects an economy that permanently has less (greater) demand strength relative to the base case, because cash flow/service flow continually and permanently reverts to the decreased (increased) the long-run equilibrium. Not surprisingly, as $R_0^L = \bar{R}_L$ increases, completed unit prices and land values increase due to the stronger economic demand. Expected time to development also decreases with a stronger economy, reducing speculative land development timing risk and expected returns.

Panel B of Table (4) shows the results from varying the initial reservation service flow, but it keeps the long-run reservation service flow at the base case value of $\bar{R}_L = 1$. For cases where R_0^L is less (more) the base case long-run value, this reflects an economy that has less (greater) demand strength relative to the base case, but this is temporary (cyclical) because service flows will revert back to the long-run equilibrium. As R_0^L increases, equilibrium completed unit prices stay the same, since the economy is only temporarily out of equilibrium, and land values increase with a stronger economy, albeit not as much as if the strength were permanent. Expected time to development also decreases sharply as developers construct while the economy is strong. With a stronger economy, time to development and the risk of this timing decreases and this is shown is lower expected returns.

We now turn our attention to varying the volatility of the local and reservation service flow parameters and the correlation between them. Panel A of Table (5) shows that as volatility increases, values for completed units increase as the option to wait for higher prices before development increases; thus, expected times to development also increase. Because the option to wait is embedded in land values, these also increase with volatility. Increased volatility translates into higher presales needed to commence development to mitigate an increase in absorption risk. Interestingly, expected returns on speculative land decreases with increases in volatility. This effect is due to the decrease in leverage in the land option from the higher completed project values needed to induce construction.

Panel B of Table (5) shows that as correlation increases, the volatility of the service flow ratio decreases and the option to wait to develop land decreases. This causes development to occur sooner and at lower prices. The leverage effect from decreased completed unit values increases expected returns as correlation increases. It is interesting to note that speculative land values are not greatly affected across the range of correlations and volatilities shown due to offsetting effects of leverage and volatility of the land option.

Table 4

Summary of Model Outputs as a Function of $R_0^L = \bar{R}_i$

Panel A: Permanent Economic Adjustment										
Column	A	B	C	D	E	F	G	H	I	J
$R_0^L = \bar{R}_i$	Equilibrium Service Flow $P_0^L = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
0.90	1.169	17.29	2.65	131	23.17	38.40	13.88%	63%	1.46	4.12
0.95	1.170	17.30	6.20	98	20.73	30.79	11.12%	44%	2.01	4.49
1.00	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
1.05	1.199	17.73	18.16	39	15.73	40.13	14.14%	31%	3.16	3.48
1.10	1.230	18.19	26.08	21	13.50	46.12	15.85%	31%	3.58	3.07
1.15	1.271	18.80	34.87	11	11.62	56.49	18.78%	25%	3.84	3.12
1.20	1.320	19.52	44.25	7	10.22	67.32	21.56%	25%	3.95	3.04
1.25	1.372	20.29	53.93	5	9.22	78.37	24.14%	25%	4.00	3.00
Panel B: Cyclical Economic Adjustment										
Adjust R_0^L $\bar{R}_i = 1$	Equilibrium Service Flow $P_0^L = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
0.75	1.18	17.45	0.17	166	26.51	29.64	10.62%	63%	2.59	3.86
0.80	1.18	17.45	0.96	149	24.81	34.28	12.28%	63%	2.59	3.86
0.85	1.18	17.45	2.36	132	23.06	33.20	11.89%	56%	2.59	3.86
0.90	1.18	17.45	4.53	112	21.31	38.09	13.64%	56%	2.59	3.86
0.95	1.18	17.45	7.55	90	19.68	33.67	12.06%	44%	2.59	3.86
1.00	1.18	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
1.05	1.18	17.45	16.06	43	16.72	37.04	13.27%	31%	2.59	3.86
1.10	1.18	17.45	21.15	23	15.18	39.40	14.11%	25%	2.59	3.86
1.15	1.18	17.45	26.24	8	13.06	41.93	0.00%	19%	2.59	3.86
1.20	1.18	17.45	30.84	0	Immediate	39.73	14.23%	0%	2.59	3.86
1.25	1.18	17.45	34.58	0	Construction	45.08	16.15%	0%	2.59	3.86

Base Case outputs are shown horizontally in bold.

Columns A and B: The equilibrium local service flow maps into the equilibrium completed building value using equation (15).

Column C: The value of speculative land represents the value of land without presales.

Columns D and E: The expected time to development is computed as the expected number of months before construction will begin on speculative land.

Expected return (IRR) is computed based on the expected cash flows from an investment in speculative land from time zero until construction commences.

Column F and G: The value of land ripe for development represents land with the presales necessary to justify commencement of construction. Land value that is ripe for development is shown as a percentage of the completed project value.

Columns H, I and J: The percentage of units to be presold is the amount needed to commence construction, which then maps into equilibrium level of absorption of units per year and the expected years to fully absorb the remaining space post construction.

Table 5
Summary of Model Outputs as a Function of Volatility and Correlation

Panel A: Volatility in Local Service Flows										
Column	A	B	C	D	E	F	G	H	I	J
$\sigma_{P^L} = \sigma_{R^L}$	Equilibrium Service Flow $P_0^L = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
.10	1.140	16.86	11.04	47	20.08	30.31	11.24%	31%	3.19	3.45
.125	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
.15	1.234	18.25	12.45	81	16.18	41.93	14.36%	44%	1.98	4.54

Panel B: Correlation in Local Service Flows										
ρ_L	Equilibrium Service Flow $P_0^L = \bar{P}_L$	Current Value of a Completed Building Unit	Value of Speculative Land	Expected Time to Development (in months)	Expected Return (IRR)	Value of Land that is Ripe for Development	Land as a Percentage of Completed Value	Percentage Units to be Presold Before Construction	Unit Absorption Rate Post Construction	Expected Absorption (in years)
0.00	1.222	18.07	12.12	78	16.62	41.41	14.32%	44%	2.10	4.28
0.25	1.180	17.45	11.43	66	18.17	35.07	12.56%	38%	2.59	3.86
0.50	1.143	16.90	11.13	49	19.83	30.35	11.22%	31%	3.14	3.50

Base Case outputs are shown horizontally in bold.

Columns A and B: The equilibrium local service flow maps into the equilibrium completed building value using equation (15).

Column C: The value of speculative land represents the value of land without presales.

Columns D and E: The expected time to development is computed as the expected number of months before construction will begin on speculative land. Expected return (IRR) is computed based on the expected cash flows from an investment in speculative land from time zero until construction commences.

Column F and G: The value of land ripe for development represents land with the presales necessary to justify commencement of construction. Land value that is ripe for development is shown as a percentage of the completed project value.

Columns H, I and J: The percentage of units to be presold is the amount needed to commence construction, which then maps into equilibrium level of absorption of units per year and the expected years to fully absorb the remaining space post construction.

As the parameters of the model are varied, we find that expected returns on speculative land investment generally range from 12% to 25%, and the level of return is directly related to the degree of leverage inherent in the land option, the elasticity/volatility of demand and the expected time to development. Absorption times to sell or lease any remaining units that are not presold run 3 to 4 years on average, including the construction time. Absorption levels and times are not drastically affected across different parameter sets as completed project prices adjust in equilibrium to mitigate changes in absorption. Land values range from 12% to 25% of completed project value. Finally, presales needed to commence construction are 33% to 60% and increase with the level of risk inherent in the land. These results are all in line with empirical results and investor surveys.

4. Conclusion

We have developed a model that generalizes and extends the standard real options framework by combining project cash flow uncertainty with a general economic variable that determines tenants/buyers demand for space through an aggregate reservation price. This economic uncertainty variable reflects the actualities of uncertain absorption and lumpy demand for space over time, and avoids the assumption that tenants are continuously present to lease (buy) space. Tenants arrive and depart according to a Poisson process with a stochastic intensity that is a function of both the current market lease price and the aggregate reservation price.

We formulate the model as an optimal switching problem from stochastic impulse control where switching corresponds to starting construction, and succeed in implementing a numerical scheme to solve for the value function and the optimal time to

begin construction. We are also able to determine the equilibrium land value including the developer's embedded options, and the expected return to the development project, among other results. Results from the model with respect to land and completed project values, absorption rates, presales needed to commence construction, and expected returns on land investment are generally in line with empirical results and investor surveys.

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Appendix A: Technical Results

In the first part of this appendix, we show that the model in this paper can be placed in the framework of stochastic impulse control models and establish that the verification theorem for general stochastic impulse control problems for Itô-Lévy processes proved in Øksendal and Sulem (2005) (Theorem 6.2, p. 83) may be applied to our problem providing sufficient conditions for the optimality of a solution.

Let \mathcal{B}_3 denote the collection of Borel sets $U \subset \mathbb{R}$ for which $0 \notin \bar{U}$.

Define the vector process for the prices

$$d\mathbf{X}(t) = d \begin{bmatrix} P_t \\ R_t \end{bmatrix} = \begin{bmatrix} \eta_P (\bar{P} e^{(\alpha_P - \alpha_M + r)t} - P_t) \\ \eta_R (\bar{R} e^{(\alpha_R - \alpha_M + r)t} - R_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_P P_t & 0 \\ 0 & \sigma_R R_t \end{bmatrix} d\mathbf{Z}(t)$$

where $\mathbf{Z}(t) = \begin{bmatrix} z_P(t) \\ z_R(t) \end{bmatrix}$. Let $q_i(t)$, $i = A, D$ be doubly stochastic Poisson processes with

respective intensities as defined in equations (16) and (18). It can be shown that these intensities are nonexplosive in that for $i = A, D$,

$$\int_0^t \lambda_i(s) ds < \infty \quad P - \text{almost surely, } t \geq 0.$$

Now we define random measures N_A and N_D by

$$N_i(t, U) = \sum_{0 < s \leq t} I_U(q_i(s) - q_i(s-)),$$

for $i = A, D$. Let $A(t)$ and $D(t)$ be processes that count total actual arrivals and total actual departures by time t . We combine $A(t)$ and $D(t)$ into a single process, $N(t) = A(t) - D(t)$, which represents total units under contract at time t . In terms of the marked point processes, we can write

$$A(t) = \int_{(0,t] \times \mathbb{R}^+} I(N(s-) + z \leq N^*) z N_A(ds, dz)$$

and

$$D(t) = \int_{(0,t] \times \mathbb{R}^+} I(N(s-) - z \leq 0) I(s < \tau) z N_D(ds, dz).$$

where τ is the switching time (construction start time). In the model of the paper, we only consider sales of whole units, i.e. $N(t) \in \mathbb{Z} \cap [0, N]$. However, here we start with the more general case of $N(t) \in [0, N]$, and later specialize to the case with $N(t) \in \mathbb{Z} \cap [0, N]$.

We also need to add an additional state to keep track of construction progress. Let $C(t)$ be the proportion of construction completed by time t . That is,

$$C(t) = \int_0^t \frac{1}{T_C} I(\tau \leq s \leq \tau + T_C) ds.$$

We can write the state process in vector form as

$$\mathbf{Y}(t) = \begin{bmatrix} t \\ P(t) \\ R(t) \\ N(t) \\ C(t) \\ \xi(t) \end{bmatrix} \in (\mathbb{R}^+)^3 \times [0, N^*] \times [0, 1] \times \{0, 1\}$$

where $\xi(t) \in \{0, 1\}$ indicates whether or not construction has begun.

Define the infinitesimal generator, \mathcal{A} , of $\mathbf{Y}(t)$ corresponding to $\tau_1 = \infty$ (starting with $\xi = 0$ and never switching) by

$$\begin{aligned} \mathcal{A}h(\mathbf{y}) &= \frac{\partial h}{\partial t} + \sum_{i=1}^2 a_i(t, \mathbf{x}) \frac{\partial h}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^2 (bb^T)_{ij}(t, \mathbf{x}) \frac{\partial^2 h}{\partial x_i \partial x_j} \\ &\quad + \sum_{j=A,D} \int_{\mathbb{R}} [h(\mathbf{y} + \gamma^{(j)}(z)) + h(\mathbf{y})] \lambda_j(t, dz) \end{aligned}$$

where $\mathbf{y} = (t, \mathbf{x}, n, c, \xi)$, and

$$\gamma^{(j)}(z) = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ 0 \\ z \\ 0 \end{bmatrix} & \text{if } j = A \text{ and} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -z \\ 0 \end{bmatrix} & \text{if } j = D. \end{cases}$$

The optimal switching problem for vector Itô processes is solved as a stochastic impulse control problem in Brekke and Øksendal (1994). Stochastic impulse control theory for jump diffusions specified as Itô-Lévy processes is developed in Øksendal and Sulem (2005). However, $A(t)$ and $D(t)$ are not Lévy processes as their time increments are neither independent nor stationary. Nevertheless, doubly stochastic Poisson processes are Markov processes and thus satisfy the Dynkin formula, which is really the crucial requirement for stochastic control theory. In particular, for a function h in the domain of \mathcal{A} , and given a set $D \subset \mathbb{R}^5$ such that the first exit time

$$\tau_D = \inf \{t > 0 \mid Y(t) \notin D\},$$

satisfies $E^y[\tau_D] < \infty$, we have

$$E^y [h(Y(\tau))] = h(y) + E^y \left[\int_0^{\tau_D} \mathcal{A}h(Y(s)) ds \right].$$

In the context of our model, this is the version of the Dynkin formula that Brekke and Øksendal (1994) require to hold for the state process.

Define the switching cost function by

$$H(\tau, \mathbf{x}) = \int_{\tau}^{\tau+T_c} K_0 e^{\alpha_K(s-\tau)} ds, \quad (1)$$

the time τ present value of construction costs from τ to $\tau + T_c$. Let $\Lambda(t)$ denote the rate at which carrying costs are incurred for holding unsold units. The expected present value of total profit

using switching time τ is given by

$$\begin{aligned}
J^\tau(\mathbf{y}) = E^y & \left[w \int_t^{\tau+T_c} \int_0^\infty e^{-r(s-t)} I(N(s) < N^*) z N_A(ds, dz) \right. \\
& + e^{-r(\tau+T_c-t)} \left[N(\tau+T_c)(P(\tau+T_c) - w) I(t \leq \tau+T_c) + \tilde{V}(\mathbf{Y}(\tau+T_c)) \right] \\
& \left. - \int_t^{\tau+T_c} e^{-r(s-t)} N^* \Lambda(s) I(s < \tau+T_c) ds - e^{-r(\tau-t)} H(\mathbf{Y}(\tau-)) \right]
\end{aligned}$$

where

$$\tilde{V}(\mathbf{Y}(t)) = E^y \left[\int_t^{t+\tau_0(\mathbf{y})} \int_0^{N^*} e^{-r(s-t)} P(s) z N_A(ds, dz) - \int_t^{t+\tau_0(\mathbf{y})} e^{-r(s-t)} (N^* - N(s)) \Lambda(s) ds \right]$$

with

$$\tau_0(\mathbf{y}) = \inf \{ s > 0; N(t+s) \geq N^* \}.$$

That is, $\tilde{V}(\mathbf{Y}(t))$ is the expected present value of the unsold units minus carrying costs at the time of construction completion given the current state at that time.

The optimal switching problem is to find

$$V(\mathbf{y}) = \sup_{\tau \in \mathcal{T}} J^\tau(\mathbf{y})$$

where \mathcal{T} is the collection of admissible stopping (switching) times and, if possible, the optimal switching time τ_1 for which

$$V(\mathbf{y}) = J^{\tau_1}(\mathbf{y}).$$

Immediately after switching, $V(\mathbf{y})$ is given by

$$\begin{aligned}
V(\mathbf{y}) = E^y & \left[w \int_t^{t+T_c} \int_0^\infty e^{-r(s-t)} I(N(s) < N^*) z N_A(ds, dz) - \int_t^{t+T_c} e^{-r(s-t)} N^* \Lambda(s) ds \right. \\
& \left. + e^{-rT_c} \left[N(t+T_c)P(t+T_c) + \tilde{V}(\mathbf{Y}(t+T_c)) \right] \right]
\end{aligned}$$

and before the switching times τ ,

$$V(\mathbf{y}) \geq E^{\mathbf{y}} \left[w \int_t^{t+T_c} \int_0^\infty e^{-r(s-t)} I(N(s) < N^*) z N_A(ds, dz) - \int_t^\tau e^{-r(s-t)} N^* \Lambda(s) ds \right. \\ \left. - e^{-r(\tau-t)} H(Y(\tau-)) I(t < \tau) + e^{-r(\tau-t)} V(Y(\tau)) \right].$$

If an optimal switching time exists, then

$$V(\mathbf{y}) = \sup_\tau E^{\mathbf{y}} \left[w \int_t^{t+T_c} \int_0^\infty e^{-r(s-t)} I(N(s) < N^*) z N_A(ds, dz) - \int_t^\tau e^{-r(s-t)} N^* \Lambda(s) ds \right. \\ \left. - e^{-r(\tau-t)} H(\mathbf{Y}(\tau-)) I(t < \tau) + e^{-r(\tau-t)} V(\mathbf{Y}(\tau)) \right].$$

Define

$$\mathcal{S} = \left\{ (t, p, r, n, c, \xi) \in (\mathbb{R}^+)^3 \times [0, N^*] \times [0, 1] \times \{0, 1\} \mid t < K, n < N^*, c < 1 \right\},$$

and let $\tau_{\mathcal{S}}$ be the first exit time from \mathcal{S} . Let $f : \mathcal{S} \rightarrow \mathbb{R}$ and suppose that $f(\mathbf{y})$ is the profit per time unit when in state \mathbf{y} , so that

$$E^{\mathbf{y}} \left[\int_t^{\tau+T_c} f(\mathbf{Y}(s)) ds \right] \\ = E^{\mathbf{y}} \left[w \int_t^{\tau+T_c} \int_0^\infty e^{-r(s-t)} I(N(s) < N^*) z N_A(ds, dz) - \int_t^{\tau+T_c} e^{-r(s-t)} N^* \Lambda(s) I(s < \tau + T_c) ds \right].$$

Let $g : \partial\bar{\mathcal{S}} \rightarrow \mathbb{R}$ be the “bequest” function, so that

$$g(\mathbf{Y}(\tau_{\mathcal{S}})) = e^{-rT_c} \left[N(\tau_{\mathcal{S}}) P(\tau_{\mathcal{S}}) + \tilde{V}(\mathbf{Y}(\tau_{\mathcal{S}})) \right].$$

Define the switching operator M on \mathcal{B} , the family of Borel measurable functions on

$(\mathbb{R}^+)^3 \times [0, N^*] \times [0, 1] \times \{0, 1\}$, by

$$Mh(t, \mathbf{x}, n, c, \xi) = h(t, \mathbf{x}, n, c, 1) - H(t, \mathbf{x}).$$

Verification Theorem. Suppose there exists $\phi : \mathcal{S} \rightarrow \mathbb{R}$ such that

(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$

(ii) $\phi \geq M\phi$ on \mathcal{S}

Define the continuation region as $\mathcal{C} = \{y \in \mathcal{S} \mid \phi(y) > M\phi(y)\}$. Furthermore, if

$$(iii) \ E^y \left[\int_0^{\tau_s} I(\mathbf{Y}(t) \in \partial\mathcal{C}) \right] = 0 \text{ for all } \mathbf{y} \in \mathcal{S}$$

(iv) $\partial\mathcal{C}$ is a Lipschitz surface (i.e. locally the graph of a Lipschitz continuous function)

(v) $\phi \in C^2(\mathcal{S} \setminus \partial\mathcal{C})$ with locally bounded derivatives near $\partial\mathcal{C}$

(vi) $A\phi + f \leq 0$ on $\mathcal{S} \setminus \partial\mathcal{C}$

(vii) $\phi(\mathbf{Y}(t)) \rightarrow g(\mathbf{Y}(\tau_s))I(\tau_s < \infty)$ as $t \rightarrow \tau_s -$ almost surely, for all $\mathbf{y} \in \mathcal{S}$

(viii) $\{\phi^-(\mathbf{Y}(\tau))\}_{\tau \in \mathcal{T}}$ is uniformly integrable for all $\mathbf{y} \in \mathcal{S}$

Then $\phi(\mathbf{y}) \geq V(\mathbf{y})$. Moreover, if it is also the case that

(ix) $A\phi + f = 0$ in \mathcal{C}

we have that, if $\tau = \inf\{t > 0 \mid \phi(\mathbf{Y}(t)) = M\phi(\mathbf{Y}(t))\}$, then $\phi(\mathbf{y}) = J^\tau(\mathbf{y}) = V(\mathbf{y})$.

Note that (ii), (vi), and (ix) can be combined into the single condition

$$\max(A\phi(\mathbf{y}) + f(\mathbf{y}), M\phi(\mathbf{y}) - \phi(\mathbf{y})) = 0, \text{ for all } \mathbf{y} \in \mathcal{S}$$

with boundary values

$$\phi(\mathbf{y}) = g(\mathbf{y}), \text{ for all } \mathbf{y} \in \partial\mathcal{S}.$$

Now we can specialize to the case with $N(t) \in \mathbb{Z} \cap [0, N]$. Assume that for $i = A, D$, the intensity kernel $\lambda_t^i(dz)$ is of the form $\lambda_t^i(dz) = \lambda_t^i \Phi(dz)$, where $\Phi(\cdot)$ is a Gaussian distribution function with mean 1 and standard deviation σ_ϕ . So changes in the number of units under contract upon tenant arrival and departure are identically distributed as $\mathcal{N}(1, \sigma_\phi)$. As $\sigma_\phi \rightarrow 0$, the verification theorem remains valid after making the obvious changes to the continuity and smoothness conditions.

In the remainder of this appendix, we present a sketch of the proof that the solution obtained from our numerical scheme is consistent with the continuous-time solution of the stochastic impulse control problem.

First, Nelson and Ramaswamy (1990) show that their approximation converges weakly to the correct diffusion, i.e. $X^\delta(\cdot) \Rightarrow X(\cdot)$. Furthermore, it can be shown that $\tau^\delta \Rightarrow \tau$. Let $D[0, K]$ denote the space of \mathbb{R} -values functions on $[0, K]$ that are right continuous and have left limits.

For $(\varphi, T) \in D[0, K] \times [0, K]$, the mapping

$$(\varphi(\cdot), T) \rightarrow \int_t^T f(\varphi(s)) ds + g(\varphi(s))$$

is continuous with probability 1 when the probability measure is that induced by the distribution of $(X(\cdot), \tau)$. Thus, by the continuous mapping theorem,

$$\int_t^{\tau_s^\delta} f(\mathbf{Y}^\delta(s)) ds + g(\mathbf{Y}^\delta(\tau_s^\delta)) \Rightarrow \int_t^{\tau_s} f(\mathbf{Y}(s)) ds + g(\mathbf{Y}(\tau_s)).$$

Since

$$\left\{ \int_t^{\tau_s^\delta} f(\mathbf{Y}^\delta(s)) ds + g(\mathbf{Y}^\delta(\tau_s^\delta)) \right\}_\delta$$

are uniformly integrable random variables, we have that

$$E^y \left[\int_t^{\tau_s^\delta} f(\mathbf{Y}^\delta(s)) ds + g(\mathbf{Y}^\delta(\tau_s^\delta)) \right] \rightarrow E^y \left[\int_t^{\tau_s} f(\mathbf{Y}(s)) ds + g(\mathbf{Y}(\tau_s)) \right], \text{ for all } y \in \mathcal{S}.$$

This suffices to show that the value functions, $V^\delta(\mathbf{y})$, from the Nelson-Ramaswamy approximation converge to a solution (at least in the viscosity sense) of

$$\max(A\phi(\mathbf{y}) + f(\mathbf{y}), M\phi(\mathbf{y}) - \phi(\mathbf{y})) = 0.$$

Appendix B: Derivation of the Numerical Method

From the paper, the risky dP process is:

$$dP_t = \kappa_{P_t}' (\bar{P}_t' e^{\alpha_P t} - P_t) dt + \sigma_P P_t dz_P(t),$$

and the risk-neutral process is:

$$dP_t = \left[\kappa_{P_t}' (\bar{P}_t' e^{(\alpha_P - \alpha_M + r)t} - P_t) - P_t (\alpha_M - r) \right] dt + \sigma_P P_t dz_P(t), \text{ or equivalently}$$

$$dP_t = \left[\kappa_{P_t}' \left(\bar{P}_t' e^{(\alpha_P - \alpha_M + r)t} - P_t \left(1 - \frac{\alpha_M - r}{\kappa_{P_t}'} \right) \right) \right] dt + \sigma_P P_t dz_P(t)$$

Similarly, examining the R processes the risky process is

$$dR_t = \kappa_{R_t}' (\bar{R}_t' e^{(\alpha_R - \alpha_M + r)t} - R_t) dt + \sigma_R R_t dz_R(t),$$

and the risk-neutral

$$dR_t = \left[\kappa_{R_t}' \left(\bar{R}_t' e^{(\alpha_R - \alpha_M + r)t} - R_t \left(1 - \frac{\alpha_M - r}{\kappa_{R_t}'} \right) \right) \right] dt + \sigma_R R_t dz_R(t)$$

Now we have to set up the numerical transformations. We begin with the two risk-neutral processes:

$$dP_t = \left[\kappa_{P_t}' \left(\bar{P}_t' e^{(\alpha_P - \alpha_M + r)t} - P_t \left(1 - \frac{\alpha_M - r}{\kappa_{P_t}'} \right) \right) \right] dt + \sigma_P P_t dz_P(t)$$

and

$$dR_t = \left[\kappa_{R_t}' \left(\bar{R}_t' e^{(\alpha_R - \alpha_M + r)t} - R_t \left(1 - \frac{\alpha_M - r}{\kappa_{R_t}'} \right) \right) \right] dt + \sigma_R R_t dz_R(t).$$

For convenience let's first work with the dP process and re-write it as:

$$dP = a(P, t)dt + b(P, t)dz_P.$$

The Nelson-Ramaswamy transformation first removes non-constant volatility in the stochastic portion. We begin this process by defining a new process $G=\ln(P)$ and noting that Ito's lemma states:

$$dG = \left(\frac{\partial G}{\partial P} a(P, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P^2} b(P, t)^2 \right) dt + \frac{\partial G}{\partial P} b(P, t) dz_P.$$

The partials for dG are:

$$\frac{\partial G}{\partial P} = \frac{1}{P}; \quad \frac{\partial G}{\partial t} = 0; \quad \frac{\partial^2 G}{\partial P^2} = -\frac{1}{P^2},$$

and substituting into Ito's lemma gives us:

$$dG = \left(\frac{1}{P} a(P, t) + 0 + \frac{1}{2} \left(-\frac{1}{P^2} \right) b(P, t)^2 \right) dt + \frac{1}{P} b(P, t) dz_P.$$

So that we wind up with

$$dG = \left(\left[\kappa_{P'} \left(\frac{\bar{P}' e^{(\alpha_P - \alpha_M + r)t}}{P(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{P'}} \right) \right) \right] - \frac{1}{2} \sigma_P^2 \right) dt + \sigma_P dz_P.$$

Similarly we can derive a similar process for R, dH:

$$dH = \left(\left[\kappa_{R'} \left(\frac{\bar{R}' e^{(\alpha_R - \alpha_M + r)t}}{R(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{R'}} \right) \right) \right] - \frac{1}{2} \sigma_R^2 \right) dt + \sigma_R dz_R$$

The correlation between dz_R and dz_P is a constant ρ (defined elsewhere.) We need to remove that correlation, so we define two new processes X_1 and X_2 which we define as:

$$X_1 = \sigma_R G + \sigma_P H$$

and

$$X_2 = \sigma_R G - \sigma_P H.$$

From Ito's lemma these processes follow:

$$dX_1 = \left[\sigma_R \left(\left[\kappa_{P'} \left(\frac{\bar{P}'_t e^{(\alpha_P - \alpha_M + r)t}}{P(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{P'}} \right) \right) \right] - \frac{1}{2} \sigma_P^2 \right) + \sigma_P \left(\left[\kappa_{R'} \left(\frac{\bar{R}'_t e^{(\alpha_R - \alpha_M + r)t}}{R(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{R'}} \right) \right) \right] - \frac{1}{2} \sigma_R^2 \right) \right] dt + \sigma_P \sigma_R \sqrt{2(1 + \rho)} dz_1$$

and

$$dX_2 = \left[\sigma_R \left(\left[\kappa_{P'} \left(\frac{\bar{P}'_t e^{(\alpha_P - \alpha_M + r)t}}{P(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{P'}} \right) \right) \right] - \frac{1}{2} \sigma_P^2 \right) - \sigma_P \left(\left[\kappa_{R'} \left(\frac{\bar{R}'_t e^{(\alpha_R - \alpha_M + r)t}}{R(t)} - \left(1 - \frac{\alpha_M - r}{\kappa_{R'}} \right) \right) \right] - \frac{1}{2} \sigma_R^2 \right) \right] dt + \sigma_P \sigma_R \sqrt{2(1 - \rho)} dz_2$$

For ease of exposition, we rewrite these as

$$dX_1 = \mu^+ dt + \sigma^+ dZ_1$$

and

$$dX_2 = \mu^- dt + \sigma^- dZ_2.$$

It is straightforward to recover G and H from X_1 and X_2 at any point in time, simply note that:

$$G = \frac{X_1 + X_2}{2\sigma_R} \quad \text{and} \quad H = \frac{X_1 - X_2}{2\sigma_P}.$$

The final step, therefore, is to calculate the required transition probabilities from a node at time t to a node at time $t+1$. Hilliard, Schwartz and Tucker (1996) demonstrate one gets convergence in the process if the probabilities are set equal to:

$$p_1 = .5 - k_1 + \frac{\mu^+ \sqrt{dt}}{2\sigma^+}$$

and

$$p_2 = .5 - k_2 + \frac{\mu^- \sqrt{dt}}{2\sigma^-}.$$

At each node the values of k_1 and k_2 must be selected to insure that $0 \leq p_1 \leq 1$ and $0 \leq p_2 \leq 1$. Since there are two stochastic variables in the model, from any specific node in the lattice, there are four possible nodes that can be attained at the next time step: where both X_1 and X_2 increase, where X_1 increases but X_2 decreases, where X_1 decreases but X_2 increases, and where both X_1 and X_2 decrease.²⁴ We denote these nodes as uu, ud, du, and dd, respectively. The probability of reaching each of these nodes is given by:

$$\begin{aligned} P_{uu} &= p_1 * p_2 \\ P_{ud} &= p_1 * (1 - p_2) \\ P_{du} &= (1 - p_1) * p_2 \\ P_{dd} &= (1 - p_1) * (1 - p_2). \end{aligned}$$

Of course we not only want to track the risk-neutral probabilities, but also we want to be able to work with the real or risky probabilities. From above, we note that the risky processes follow:

$$dP_t = \kappa_{P_t}' (\bar{P}_t' e^{\alpha P_t} - P_t) dt + \sigma_P P_t dz_P(t),$$

and

$$dR_t = \kappa_{R_t}' (\bar{R}_t' e^{\alpha R_t} - R_t) dt + \sigma_R R_t dz_R(t), .$$

We then have to go through the Ito transformations. We can once again re-write dP_t as:

$$dP = A(P, t) dt + b(P, t) dz_P$$

Except that we use the upper-case alpha to denote the different drift term from the risk-neutral process. If we define a function $G = \ln(P)$ (again in the risky context) we from Ito's lemma:

$$dG = \left(\frac{\partial G}{\partial P} A(P, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P^2} b(P, t)^2 \right) dt + \frac{\partial G}{\partial P} b(P, t) dz_P .$$

The partials for dG are:

²⁴ As Hilliard, Kau, and Slawson (1995) discuss, to insure well-behaved probabilities it is sometimes necessary for k_1 and k_2 to take on values such that X_1 or X_2 will increase or decrease in both their "up" and "down" states. When that happens, however, the "up" version of either X_1 or X_2 is still greater than the "down" version of X_1 or X_2 .

$$\frac{\partial G}{\partial P} = \frac{1}{P}; \quad \frac{\partial G}{\partial t} = 0; \quad \frac{\partial^2 G}{\partial P^2} = -\frac{1}{P^2},$$

and substituting into Ito's lemma gives us:

$$dG = \left(\frac{1}{P} a(P, t) + 0 + \frac{1}{2} \left(-\frac{1}{P^2} \right) b(P, t)^2 \right) dt + \frac{1}{P} b(P, t) dz_P.$$

So that we wind up with

$$dG = \left(\left[\kappa_{P'} \left(\frac{\bar{P}' e^{\alpha_P t}}{P(t)} - 1 \right) \right] - \frac{1}{2} \sigma_P^2 \right) dt + \sigma_P dz_P.$$

Using the same logic for the risky R process, we define H=ln(R) and we wind up with

$$dH = \left(\left[\kappa_{R'} \left(\frac{\bar{R}' e^{\alpha_R t}}{R(t)} - 1 \right) \right] - \frac{1}{2} \sigma_R^2 \right) dt + \sigma_R dz_R$$

We can now go through a remove the instantaneous correlation between dG and dH by defining two new processes Y₁ and Y₂

$$Y_1 = \sigma_R G + \sigma_P H$$

and

$$Y_2 = \sigma_R G - \sigma_P H.$$

From Ito's lemma these processes follow:

$$dY_1 = \left[\sigma_R \left(\left[\kappa_{P'} \left(\frac{\bar{P}' e^{\alpha_P t}}{P(t)} - 1 \right) \right] - \frac{1}{2} \sigma_P^2 \right) + \sigma_P \left(\left[\kappa_{R'} \left(\frac{\bar{R}' e^{\alpha_R t}}{R(t)} - 1 \right) \right] - \frac{1}{2} \sigma_R^2 \right) \right] dt + \sigma_P \sigma_R \sqrt{2(1+\rho)} dz_1$$

and

$$dY_2 = \left[\sigma_R \left(\left[\kappa_{P'} \left(\frac{\bar{P}' e^{\alpha_P t}}{P(t)} - 1 \right) \right] - \frac{1}{2} \sigma_P^2 \right) - \sigma_P \left(\left[\kappa_{R'} \left(\frac{\bar{R}' e^{\alpha_R t}}{R(t)} - 1 \right) \right] - \frac{1}{2} \sigma_R^2 \right) \right] dt + \sigma_P \sigma_R \sqrt{2(1-\rho)} dz_2$$

For ease of exposition, we rewrite these as

$$dY_1 = \mu^+ dt + \sigma^+ dZ_1$$

and

$$dY_2 = \mu^- dt + \sigma^- dZ_2.$$

Recovery of G and H values is exactly analogous the process used in equation (15):

$$G = \frac{Y_1 + Y_2}{2\sigma_R} \text{ and, } H = \frac{Y_1 - Y_2}{2\sigma_P}.$$

Similarly, we will get convergence in the process if the risky-probabilities are set equal to:

$$p_1^{\text{Real}} = .5 - k_1 + \frac{\mu^+ \sqrt{dt}}{2\sigma^+}$$

and

$$p_2^{\text{Real}} = .5 - k_2 + \frac{\mu^- \sqrt{dt}}{2\sigma^-}.$$

Where once again the values of k_1 and k_2 are selected to insure valid values for p_1^{Real} and p_2^{Real} .

The risky transition probabilities, therefore, are given by:

$$\begin{aligned} P_{uu}^{\text{Real}} &= p_1^{\text{Real}} * p_2^{\text{Real}} \\ P_{ud}^{\text{Real}} &= p_1^{\text{Real}} * (1 - p_2^{\text{Real}}) \\ P_{du}^{\text{Real}} &= (1 - p_1^{\text{Real}}) * p_2^{\text{Real}} \\ P_{dd}^{\text{Real}} &= (1 - p_1^{\text{Real}}) * (1 - p_2^{\text{Real}}). \end{aligned}$$

These risk-neutral probabilities can then be used at an arbitrary node t,i,j to determine the

risk-neutral expected value of a variable at time $t+1$.